

Lead Compensation

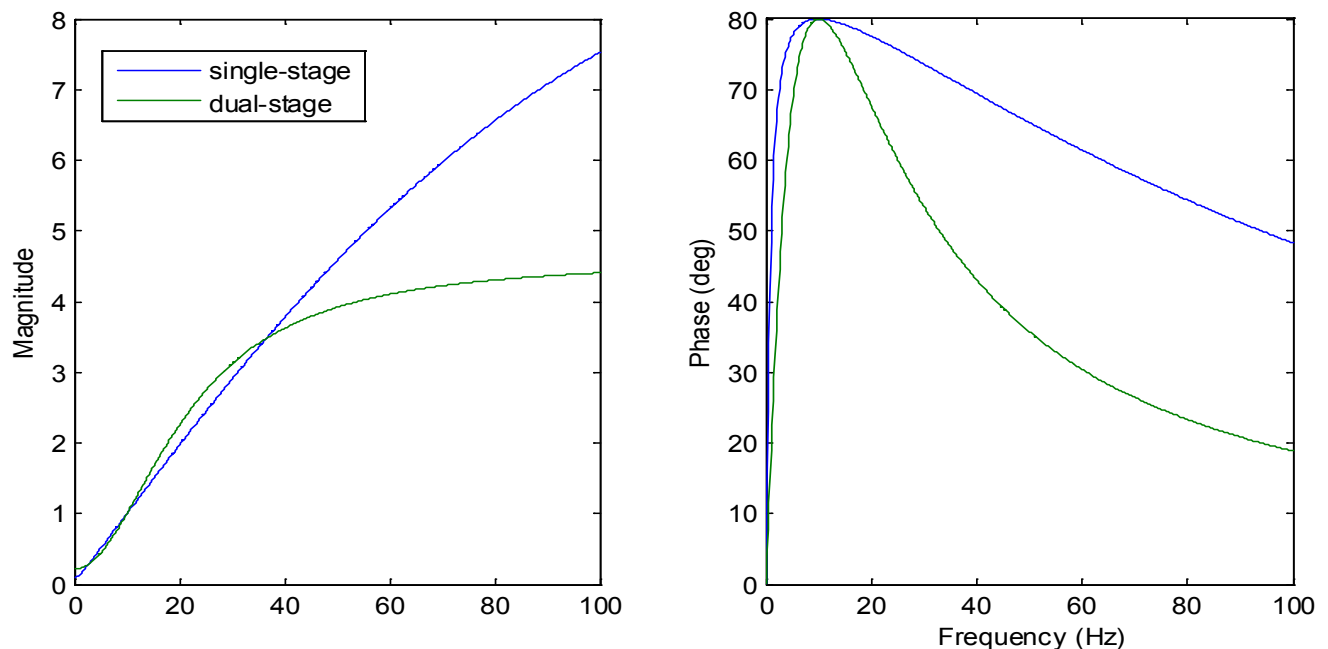
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This report examines lead compensation, which is used to mitigate the effect of delay in a closed loop control system. First to be discussed is the traditional lead compensator consisting of a real pole and a real zero, followed by an improved lead compensator having complex poles and zeros. This work is based on the reference below.

Reference: Messner, et al., *Lead and Lag Compensators with Complex Poles and Zeros*, IEEE Control Systems Magazine, February 2007, pp. 44-54.

Real Pole/Zero Lead Compensation

The lead compensator most often seen in textbooks and the literature is a first-order network consisting of a real pole and a real zero. To obtain phase lead, the frequency of the zero must be lower than that of the pole, resulting in an upward break of the phase frequency response. The magnitude of the network also increases with frequency, which is an undesirable effect in a closed-loop control system because it erodes gain margin. A first-order lead compensator is theoretically limited to 90 degrees of lead, but practical considerations may reduce this limit to a lower value. Additional lead can be obtained by cascading first-order lead networks. In the reference above, the authors limit the maximum lead of a single network to 60 degrees, perhaps because in their opinion the gain increase becomes too great. They suggest breaking the total lead into a cascade of lower-order lead networks: *"The high frequency gain of a single lead compensator with phase lead greater than 60 degrees is significantly greater than the high frequency gain of n cascaded lead compensators each contributing $1/n$ of the same total phase lead."* The veracity of this statement is confirmed in the following frequency response plot for an 80 degree lead compensator, implemented both as a single 80 degree lead compensator, and as a cascade of two 40 degree lead compensators:



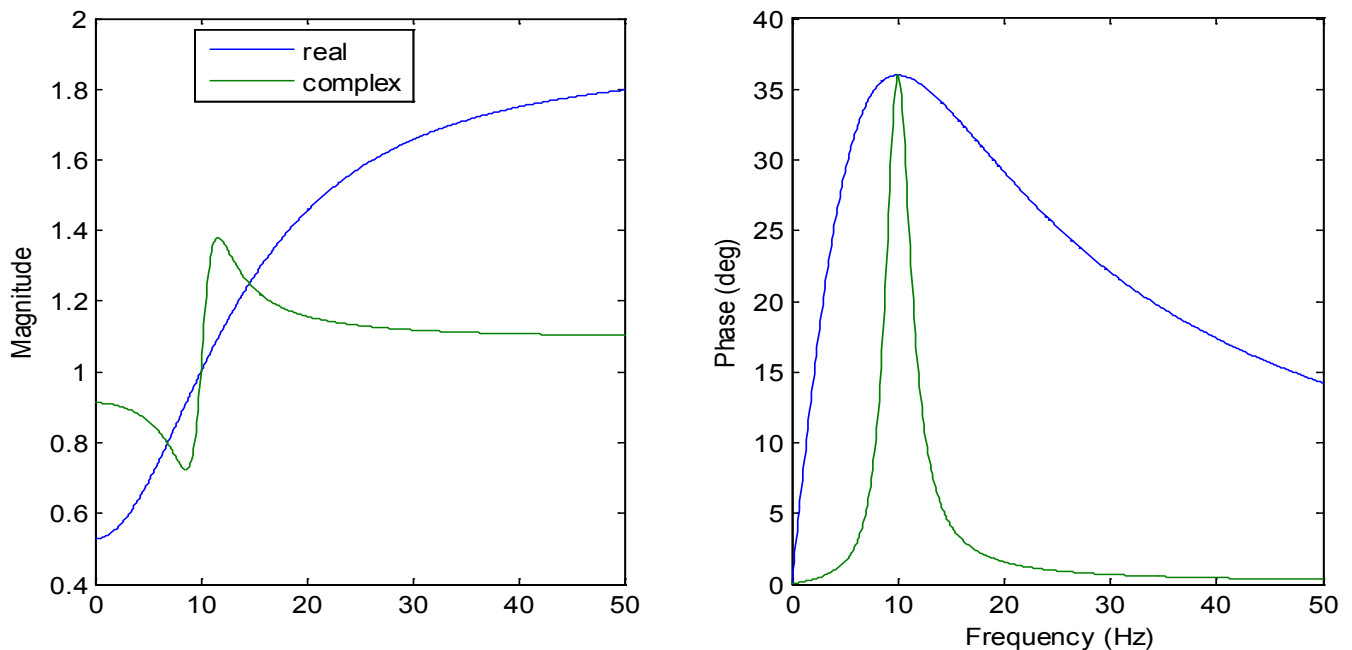
Note in the figure that the dual-stage lead compensator has less gain increase, which is desirable, but it also has a narrower bandwidth of phase lead, which may or may not be desirable. If phase lead is required only over reduced bandwidth, then the multi-stage approach to lead compensation is clearly superior.

This example highlights an important principle: Along with broad phase lead bandwidth comes increased high frequency gain, which erodes gain margin. To reduce high frequency gain, reduce the phase lead bandwidth. With real pole/zero lead compensators, this can be done by splitting the lead into multiple stages. Another way to accomplish this is to use a lead compensator consisting of complex poles and zeros, discussed next.

Complex Pole/Zero Lead Compensation

A lead compensator with complex poles and zeros allows tradeoff between phase lead bandwidth and high frequency gain: gain can be reduced by reducing lead bandwidth. The additional design parameter that permits this tradeoff is pole/zero damping, which in real pole/zero lead compensators is fixed at unity.

Below is an example of a 36 degrees @ 10Hz lead compensator implemented in both real and complex form, the real form consisting of two 18 degree lead compensators in series. Note that the complex compensator has much narrower lead bandwidth and consequently has much lower high frequency gain.



Both the poles and zeros of the complex lead compensator share the same damping ratio that is less than unity. The smaller the damping ratio, the narrower the phase lead bandwidth and the smaller the high frequency gain. When the damping ratio is unity, the compensator reverts to a second-order lead compensator with real poles and zeros.

Design Equations: First-Order Lead Compensator

Note: This section is included only for completeness. As stated previously, it is better use split the lead into two first-order networks rather than one. A second-order lead compensator, which is discussed in the next section, is preferred.

The transfer function of a first-order lead compensator having unity gain and phase lead of ϕ_o at ω_o is

$$G_{LEAD}(s) = \sqrt{\frac{\omega_{POLE}}{\omega_{ZERO}}} \cdot \frac{s + \omega_{ZERO}}{s + \omega_{POLE}} \quad (\text{eqn. 1, p. 44 of the reference})$$

where

$$\omega_{ZERO} = \omega_o \sqrt{\frac{1 - \sin \phi_o}{1 + \sin \phi_o}} \quad \text{zero frequency (eqn. 4, p. 45 of the reference)}$$

$$\omega_{POLE} = \omega_o^2 / \omega_{ZERO} \quad \text{pole frequency (eqn. 2, p. 44 of the reference)}$$

Solve for ω_{ZERO} , and ω_{POLE} while keeping the gain ratio less than or equal to a desired value K :

The ratio of gain increase from DC to infinite frequency, K , is given by

$$K = \frac{|G_{LEAD}(\infty)|}{|G_{LEAD}(0)|} = \frac{\left| \frac{j\infty + \omega_{ZERO}}{j\infty + \omega_{POLE}} \right|}{\left| \frac{j0 + \omega_{ZERO}}{j0 + \omega_{POLE}} \right|} = \frac{1}{\frac{\omega_{ZERO}}{\omega_{POLE}}} = \frac{\omega_{POLE}}{\omega_{ZERO}}$$

The relationship between center frequency ω_o and pole and zero frequencies ω_{POLE} and ω_{ZERO} is given by

$$\omega_o^2 = \omega_{POLE} \omega_{ZERO} \quad \text{center frequency (eqn. 2, p. 44 of the reference)}$$

Substituting $\omega_{ZERO} K$ for ω_{POLE} ,

$$\begin{aligned} \omega_o^2 &= \omega_{POLE} \omega_{ZERO} = \omega_{ZERO} \omega_{ZERO} K \\ \Rightarrow \omega_{ZERO} &= \omega_o / \sqrt{K} \end{aligned}$$

Similarly, ω_{POLE} / K for ω_{ZERO} ,

$$\begin{aligned} \omega_o^2 &= \omega_{POLE} \omega_{ZERO} = \omega_{POLE} \omega_{POLE} / K \\ \Rightarrow \omega_{POLE} &= \omega_o \sqrt{K} \end{aligned}$$

The relationship between gain ratio K and phase lead ϕ_o is derived as follows: The zero frequency ω_{ZERO} can be expressed as a function of phase lead ϕ_o as

$$\omega_{ZERO} = \omega_o \sqrt{\frac{1 - \sin \phi_o}{1 + \sin \phi_o}}$$

zero frequency (eqn. 4, p. 45 of the reference)

Since $\omega_{ZERO} = \omega_o / \sqrt{K}$,

$$K = \frac{1 + \sin \phi_o}{1 - \sin \phi_o}$$

Summary: First-Order Lead Compensator

Design equations for a first-order lead compensator having unity gain and phase lead of ϕ_o at ω_o :

$$G_{LEAD}(s) = \sqrt{\frac{\omega_{POLE}}{\omega_{ZERO}}} \cdot \frac{s + \omega_{ZERO}}{s + \omega_{POLE}} \quad \text{first-order lead compensator transfer function}$$

where

$$K = \frac{|G_{LEAD}(\infty)|}{|G_{LEAD}(0)|} = \frac{1 + \sin \phi_o}{1 - \sin \phi_o}$$

$$\omega_{ZERO} = \omega_o / \sqrt{K} \quad \text{first-order lead compensator parameters}$$

$$\omega_{POLE} = \omega_o \sqrt{K}$$

Design Equations: Second-Order Lead Compensator

The transfer function of a second-order lead compensator having unity gain and phase lead of ϕ_o at ω_o is

$$G_{LEAD}(s) = \frac{\omega_{POLE}}{\omega_{ZERO}} \cdot \frac{s^2 + 2\zeta\omega_{ZERO}s + \omega_{ZERO}^2}{s^2 + 2\zeta\omega_{POLE}s + \omega_{POLE}^2} \quad (\text{eqn. 6, p. 45 of the reference})$$

where

ζ	pole/zero damping ratio
ω_{ZERO}	zero frequency
ω_{POLE}	pole frequency

Solve for ζ , ω_{ZERO} , and ω_{POLE} while keeping the gain ratio less than or equal to a desired value K :

The ratio of gain increase from DC to infinite frequency, K , is given by

$$K = \frac{|G_{LEAD}(\infty)|}{|G_{LEAD}(0)|} = \frac{\left| \frac{(j\infty)^2 + 2\zeta\omega_{ZERO}j\infty + \omega_{ZERO}^2}{(j\infty)^2 + 2\zeta\omega_{POLE}j\infty + \omega_{POLE}^2} \right|}{\left| \frac{(j0)^2 + 2\zeta\omega_{ZERO}j0 + \omega_{ZERO}^2}{(j0)^2 + 2\zeta\omega_{POLE}j0 + \omega_{POLE}^2} \right|} = \frac{1}{\frac{\omega_{ZERO}^2}{\omega_{POLE}^2}} = \left(\frac{\omega_{POLE}}{\omega_{ZERO}} \right)^2$$

The relationship between center frequency ω_o and pole and zero frequencies ω_{POLE} and ω_{ZERO} is given by

$$\omega_o^2 = \omega_{POLE}\omega_{ZERO} \quad \text{center frequency (eqn. 2, p. 44 of the reference)}$$

Substituting $\omega_{ZERO}\sqrt{K}$ for ω_{POLE} ,

$$\begin{aligned} \omega_o^2 &= \omega_{POLE}\omega_{ZERO} = \omega_{ZERO}\omega_{ZERO}\sqrt{K} \\ \Rightarrow \omega_{ZERO} &= \omega_o / K^{1/4} \end{aligned}$$

Similarly, $\omega_{POLE}/\sqrt{K}$ for ω_{ZERO} ,

$$\begin{aligned} \omega_o^2 &= \omega_{POLE}\omega_{ZERO} = \omega_{POLE}\omega_{POLE}/\sqrt{K} \\ \Rightarrow \omega_{POLE} &= \omega_o K^{1/4} \end{aligned}$$

The pole/zero damping ratio ζ that results in gain ratio K is derived as follows: The zero frequency ω_{ZERO} can be expressed as a function of damping ratio ζ as

$$\omega_{ZERO} = \omega_o \left(-\zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1} \right) \quad \text{zero frequency (eqn. 11, p. 47 of the reference)}$$

Note that here, phase ϕ is *half* of the desired phase lead ϕ_o : $\phi = \phi_o/2$.

Inserting this expression into that of the gain ratio K ,

$$\begin{aligned}
K &= \left(\frac{\omega_{POLE}}{\omega_{ZERO}} \right)^2 = \left(\frac{\omega_o^2 / \omega_{ZERO}}{\omega_{ZERO}} \right)^2 = \left(\frac{\omega_o^2}{\omega_o^2 \left(-\zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1} \right)^2} \right)^2 \\
\Rightarrow K &= \left(\frac{1}{-\zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1}} \right)^4 = \left(\frac{\zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1}}{\left(-\zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1} \right) \left(\zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1} \right)} \right)^4 \\
\Rightarrow K &= \left(\frac{\zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1}}{- (\zeta \tan \phi)^2 - \zeta \tan \phi \sqrt{(\zeta \tan \phi)^2 + 1} + \zeta \tan \phi \sqrt{(\zeta \tan \phi)^2 + 1} + (\zeta \tan \phi)^2 + 1} \right)^4 \\
\Rightarrow K &= \left(\zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1} \right)^4 \\
\Rightarrow K^{1/4} &= \zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1} \\
\Rightarrow (K^{1/4} - \zeta \tan \phi)^2 &= (\zeta \tan \phi)^2 + 1 \\
\Rightarrow K^{1/2} - 2K^{1/4} \zeta \tan \phi + (\zeta \tan \phi)^2 &= (\zeta \tan \phi)^2 + 1 \\
\Rightarrow K^{1/2} - 1 &= 2K^{1/4} \zeta \tan \phi \\
\Rightarrow \zeta &= \frac{\sqrt{K} - 1}{2K^{1/4} \tan \phi}
\end{aligned}$$

This damping ratio must not exceed unity. If it evaluates to greater than unity for a given combination of desired K and ϕ , then clip it to unity, and compute the achieved K using

$$K = \left(\zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1} \right)^4$$

It was stated earlier that the smaller the damping ratio, the smaller the high frequency gain. This relationship can be seen clearly in the equation above, where a smaller ζ results in a smaller K .

It is useful to have a formula for the phase lead bandwidth, defined as the frequency band where the phase lead is greater than half of the maximum phase lead ϕ_o . This formula is derived below:

Starting with a formula for phase lead as a function of frequency,

$$\angle C(j\omega) = \arctan \left(\frac{4\zeta^2 \omega_o \tan \phi (\omega_o^2 + \omega^2) \omega}{(\omega_o^2 - \omega^2)^2 + 4\zeta^2 \omega_o^2 (1 - \tan^2 \phi) \omega^2} \right) \quad (\text{eqn. 17, p. 48 of the reference})$$

set the phase $\angle C(j\omega)$ to half of max phase lead $\phi = \phi_o/2$, take the tangent of both sides, collect like powers of ω , and normalize frequency to unity using $\bar{\omega} = \omega/\omega_o$:

$$\tan \phi = \frac{4\zeta^2 \omega_o \tan \phi (\omega_o^2 + \omega^2) \omega}{(\omega_o^2 - \omega^2)^2 + 4\zeta^2 \omega_o^2 (1 - \tan^2 \phi) \omega^2}$$

$$\Rightarrow (\omega_o^2 - \omega^2)^2 + 4\zeta^2 \omega_o^2 (1 - \tan^2 \phi) \omega^2 = 4\zeta^2 \omega_o (\omega_o^2 + \omega^2) \omega$$

$$\Rightarrow \omega_o^4 - 2\omega_o^2 \omega^2 + \omega^4 + 4\zeta^2 \omega_o^2 (1 - \tan^2 \phi) \omega^2 = 4\zeta^2 \omega_o^3 \omega + 4\zeta^2 \omega_o \omega^3$$

$$\Rightarrow \omega^4 - 4\zeta^2 \omega_o \omega^3 + [4\zeta^2 \omega_o^2 (1 - \tan^2 \phi) - 2\omega_o^2] \omega^2 - 4\zeta^2 \omega_o^3 \omega + \omega_o^4 = 0$$

$$\Rightarrow \frac{\omega^4}{\omega_o^4} - 4\zeta^2 \frac{\omega^3}{\omega_o^3} + [4\zeta^2 (1 - \tan^2 \phi) - 2] \frac{\omega^2}{\omega_o^2} - 4\zeta^2 \frac{\omega}{\omega_o} + 1 = 0$$

$$\Rightarrow \bar{\omega}^4 - 4\zeta^2 \bar{\omega}^3 + [4\zeta^2 (1 - \tan^2 \phi) - 2] \bar{\omega}^2 - 4\zeta^2 \bar{\omega} + 1 = 0$$

Find the roots of the above quartic polynomial: Because the phase frequency characteristic is a simple peak that begins and ends at zero, as seen in the previous plots, the above equation has only two zero crossings. Thus it can be surmised that there are two real roots and two complex roots. The real roots are the frequencies being sought; the complex roots can be ignored. Thus the above quartic polynomial in factored form is

$$(\bar{\omega}^2 - a\bar{\omega} + 1) \cdot (\bar{\omega} - b) \cdot (\bar{\omega} - c) = 0$$

which in expanded form is

$$(\bar{\omega}^2 - a\bar{\omega} + 1) \cdot (\bar{\omega}^2 - (b+c)\bar{\omega} + bc) = 0$$

$$\Rightarrow \bar{\omega}^4 - [a+b+c]\bar{\omega}^3 + [a(b+c) + bc + 1]\bar{\omega}^2 - [abc + (b+c)]\bar{\omega} + bc = 0$$

Comparing coefficients the above polynomial with those of the quartic polynomial,

$$bc = 1$$

$$\Rightarrow \bar{\omega}^4 - [a+b+c]\bar{\omega}^3 + [a(b+c) + 2]\bar{\omega}^2 - [a+b+c]\bar{\omega} + 1 = 0$$

$$a+b+c = 4\zeta^2 \quad \Rightarrow \quad a = 4\zeta^2 - (b+c)$$

$$a(b+c) + 2 = 4\zeta^2 (1 - \tan^2 \phi) - 2$$

$$\Rightarrow (4\zeta^2 - (b+c)) \cdot (b+c) = 4\zeta^2 (1 - \tan^2 \phi) - 4$$

$$\Rightarrow (4\zeta^2 - d) \cdot d = 4\zeta^2 (1 - \tan^2 \phi) - 4 \quad \text{where } d = b+c = b+1/b$$

$$\Rightarrow 0 = d^2 - 4\zeta^2 d + 4\zeta^2(1 - \tan^2 \phi) - 4$$

Solving the above quadratic polynomial for $d = b + 1/b$,

$$d = \frac{4\zeta^2 \pm \sqrt{(4\zeta^2)^2 - 4(4\zeta^2(1 - \tan^2 \phi) - 4)}}{2}$$

$$\Rightarrow d = 2\zeta^2 \pm \sqrt{4\zeta^4 - (4\zeta^2(1 - \tan^2 \phi) - 4)}$$

$$\Rightarrow d = 2\zeta^2 \pm 2\sqrt{\zeta^4 + (\zeta \tan \phi)^2 - \zeta^2 + 1}$$

The quantity under the radical is always greater than unity, thus the positive root is the desired solution:

$$d = 2\zeta^2 + 2\sqrt{\zeta^4 + (\zeta \tan \phi)^2 - \zeta^2 + 1}$$

Solving $d = b + 1/b$ for normalized radian frequency b ,

$$db = b^2 + 1$$

$$\Rightarrow 0 = b^2 - db + 1$$

$$\Rightarrow b = \frac{d \pm \sqrt{d^2 - 4}}{2}$$

Choose the minus sign for b and the plus sign for c :

$$b = \frac{d - \sqrt{d^2 - 4}}{2} \quad c = \frac{d + \sqrt{d^2 - 4}}{2}$$

Note that the above normalized frequencies satisfy the condition that $bc = 1$

$$bc = \frac{d - \sqrt{d^2 - 4}}{2} \cdot \frac{d + \sqrt{d^2 - 4}}{2} = \frac{d^2 - (d^2 - 4)}{4} = \frac{4}{4} = 1$$

Move the divisor of two into $\bar{d} = d/2$:

$$b = \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - 1} = \bar{d} - \sqrt{\bar{d}^2 - 1} \quad c = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 - 1} = \bar{d} + \sqrt{\bar{d}^2 - 1}$$

$$\bar{d} = \zeta^2 + \sqrt{\zeta^4 - (\zeta \tan \phi)^2 + 1}$$

De-normalize the frequencies, and rename $\bar{d}$ as $\bar{\omega}_{LU}$:

$$\omega_L = \left(\bar{\omega}_{LU} - \sqrt{\bar{\omega}_{LU}^2 - 1} \right) \omega_o \quad \omega_U = \left(\bar{\omega}_{LU} + \sqrt{\bar{\omega}_{LU}^2 - 1} \right) \omega_o$$

$$\bar{\omega}_{LU} = \zeta^2 + \sqrt{\zeta^4 + (\zeta \tan \phi)^2 - \zeta^2 + 1}$$

The phase lead bandwidth is thus

$$\Delta\omega = \omega_U - \omega_L = \left(2\sqrt{\bar{\omega}_{LU}^2 - 1} \right) \omega_o$$

It was stated earlier that the smaller the damping ratio, the narrower the phase lead bandwidth. This relationship can be seen clearly in the equations above, where a smaller ζ results in a smaller $\bar{\omega}_{LU}$, which in turn results in a smaller $\Delta\omega$.

Summary: Second-Order Lead Compensator

Design equations for a second-order lead compensator having unity gain and phase lead of ϕ_o at ω_o :

$$G_{LEAD}(s) = \frac{\omega_{POLE}}{\omega_{ZERO}} \cdot \frac{s^2 + 2\zeta\omega_{ZERO}s + \omega_{ZERO}^2}{s^2 + 2\zeta\omega_{POLE}s + \omega_{POLE}^2} \quad \text{second-order lead compensator transfer function}$$

where

$$K = \text{desired gain ratio} = \frac{|G_{LEAD}(\infty)|}{|G_{LEAD}(0)|}$$

$$\phi = \phi_o / 2$$

$$\zeta = \min \left(\frac{\sqrt{K} - 1}{2K^{1/4} \tan \phi}, 1 \right) \quad \text{second-order lead compensator parameters}$$

$$K = \left(\zeta \tan \phi + \sqrt{(\zeta \tan \phi)^2 + 1} \right)^4$$

$$\omega_{ZERO} = \omega_o / K^{1/4}$$

$$\omega_{POLE} = \omega_o K^{1/4}$$

The resulting phase lead bandwidth, defined as defined as the frequency band where the phase lead is greater than half of the maximum phase lead ϕ_o , is given by

$$\Delta\omega = \omega_U - \omega_L = \left(2\sqrt{\bar{\omega}_{LU}^2 - 1} \right) \omega_o \quad \text{second-order lead compensator phase bandwidth}$$

where

$$\omega_U = \left(\bar{\omega}_{LU} + \sqrt{\bar{\omega}_{LU}^2 - 1} \right) \omega_o$$

$$\omega_L = \left(\bar{\omega}_{LU} - \sqrt{\bar{\omega}_{LU}^2 - 1} \right) \omega_o$$

$$\bar{\omega}_{LU} = \zeta^2 + \sqrt{\zeta^4 + (\zeta \tan \phi)^2 - \zeta^2 + 1}$$

Design Code: Second-Order Lead Compensator

The following is Matlab code that implements the design equations above. To facilitate transcription to other programming languages, no special Matlab functions are used.

The variable names of the input parameters are:

leadFreq	phase lead required at specified lead frequency (deg)
leadFreq	frequency of maximum phase lead (Hz)
gainRatio	limit on the ratio of maximum gain to minimum gain (always > 1) (if unspecified, no limit is applied)
samplePeriod	compensator sample period (sec)

The code body is:

```
fsamp = 1 / samplePeriod;
if isempty(gainRatio)
    gainRatio = Inf;
end

% return wire if lead phase or lead frequency is zero
a = 1;
b = 1;
if ~(leadPhase > 0) || ~(leadFreq > 0)
    return
end

leadPhase = leadPhase * pi/180;
maxPhase = 89.0 * pi/180;

% limit gain ratio
minGainRatio = 1.001;
maxGainRatio = 1000;
k = min(max(gainRatio, minGainRatio), maxGainRatio);

% determine compensator damping
centOmega = 2*pi*leadFreq;
phi = leadPhase / 2;
phi = min(phi, maxPhase);
tanphi = tan(phi);
damp = ((sqrt(k) - 1) / (2*sqrt(sqrt(k)))) / tanphi;
damp = min(damp, 1);
k = (damp*tanphi + sqrt((damp*tanphi)^2 + 1))^4;

% determine half-phase lead frequencies
omegaLU = damp^2 + sqrt(damp^4 + (damp * tanphi)^2 + 1 - damp^2);
flo = (omegaLU - sqrt(omegaLU^2 - 1)) * leadFreq;
fup = (omegaLU + sqrt(omegaLU^2 - 1)) * leadFreq;

% design laplace domain compensator
zeroOmega = centOmega / sqrt(sqrt(k));
poleOmega = centOmega * sqrt(sqrt(k));
scale = sqrt( ( (poleOmega^2-centOmega^2)^2 ...
                + (2*damp*poleOmega*centOmega)^2) ...
              / ( (zeroOmega^2-centOmega^2)^2 ...
                + (2*damp*zeroOmega*centOmega)^2) );
a = [1 2*damp*poleOmega poleOmega^2];
b = [1 2*damp*zeroOmega zeroOmega^2] * scale;

% transform to z domain using bilinear transformation with prewarping
if ~isempty(fsamp) && (leadPhase > 0)
```

```
omega = centOmega;
as     = a;
bs     = b;
a      = zeros(1, 3);
b      = zeros(1, 3);
c      = omega / tan(omega / (2*fsamp));
cc     = c * c;
d      = as(3) + as(2)*c + as(1)*cc;
a(1)   = 1.0;
a(2)   = 2.0*(as(3)-as(1)*cc)/d;
a(3)   = (as(3)-as(2)*c+as(1)*cc)/d;
b(1)   = (bs(3)+bs(2)*c+bs(1)*cc)/d;
b(2)   = 2.0*(bs(3)-bs(1)*cc)/d;
b(3)   = (bs(3)-bs(2)*c+bs(1)*cc)/d;
end
```