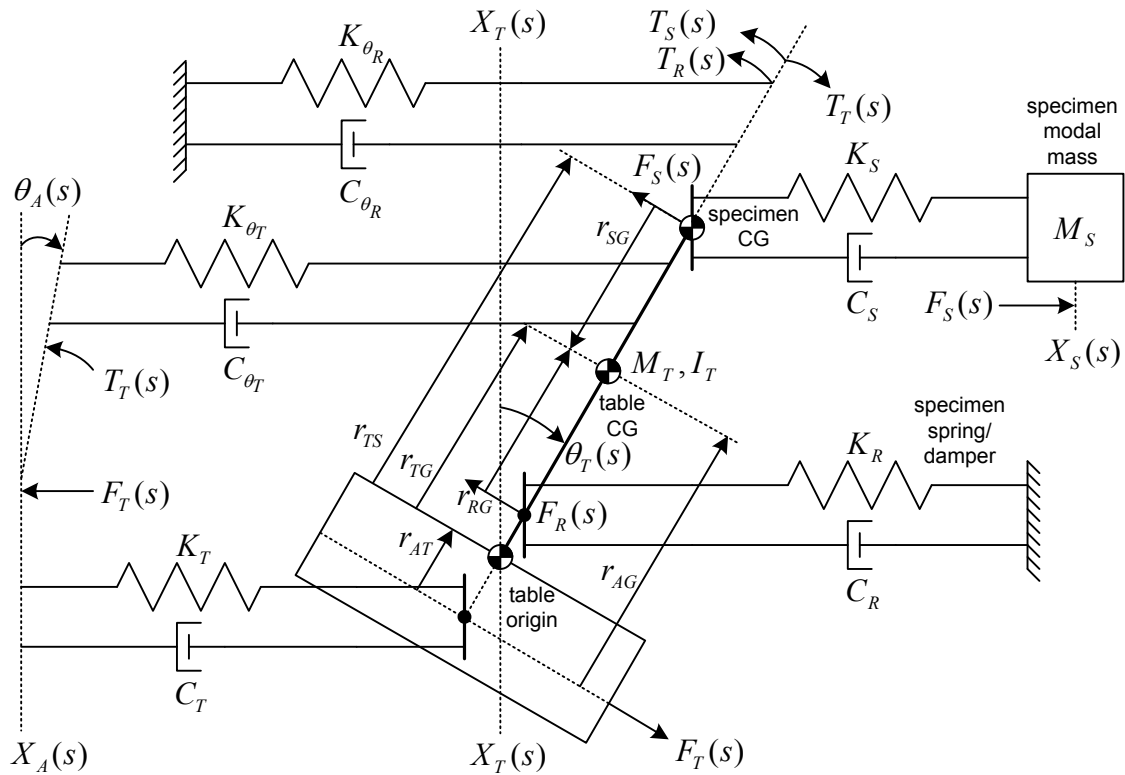


Shake Table Transfer Functions

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Variables

$X_A(s)$	Oil spring translation
$X_T(s)$	Table translation
$X_S(s)$	Specimen translation
$F_T(s)$	Translational force applied to table via translational prime mover
$F_S(s)$	Translational force applied to table via modal specimen
$F_R(s)$	Translational force applied to table spring/damper specimen
$\theta_A(s)$	Oil spring rotation
$\theta_T(s)$	Table rotation
$T_T(s)$	Rotational force applied to table via rotational prime mover
$T_S(s)$	Rotational force applied to table via modal specimen
$T_R(s)$	Rotational force applied to table via spring/damper specimen
$X_T(s)$	Table translation
M_T, C_T, K_T	Mass, damping, and spring constant of the translational prime mover
$I_T, C_{\theta_T}, K_{\theta_T}$	Inertia, damping, and spring constant of the rotational prime mover
M_S, C_S, K_S	Mass, damping, and spring constant of the modal specimen
C_R, K_R	Damping and spring constant of the translational spring/damper specimen

$C_{\theta_R}, K_{\theta_R}$	Damping and spring constant of the rotational spring/damper specimen
r_{TG}	Radius vector from table origin to table CG
r_{TA}	Radius vector from table origin to horizontal prime mover
r_{TS}	Radius vector from table origin to specimen CG
r_{AG}	Radius vector from horizontal prime mover to table CG
r_{SG}	Radius vector from specimen CG to table CG
r_{RG}	Radius vector from spring/damper to table CG

Notes

- 1) Vertical distances are signed: positive is up, negative is down.
- 2) Horizontal distances are signed: positive is right, negative is left.
- 3) Clockwise rotation is positive.
- 4) Superscripts denote frame:

G	Center of gravity (CG)
T	Table origin
(blank)	frame independent
- 5) Subscripts denote entity:

A	Prime mover (actuator)
T	Table
S	Specimen
V	Velocity
F	Force
X1, Z1, X2	Individual actuators
- 6) Radius vectors with dual subscripts have sign: $r_{PQ} = r_Q - r_P$ is positive if $r_Q > r_P$, negative if $r_Q < r_P$.
- 7) Moment equations are developed with relative positions of table origin, table, CG and specimen CG as shown in the diagram. If the relative positions are not as shown, for example if the specimen CG is below the table CG, the moment arm radius vectors, because they are signed quantities, will correctly reverse the signs of the moments.

Ultimately, the motion and force equations will be derived in terms of damping and natural frequency. The following definitions will be utilized:

Damping ratios and natural frequencies in terms of mass, inertia, damping, and spring constants

$$\frac{C_T^G}{M_T} = 2\zeta_T \omega_T$$

$$\frac{C_{\theta_T}^G}{I_T^G} = 2\zeta_{\theta_T} \omega_{\theta_T}$$

$$\frac{C_S^G}{M_S} = 2\zeta_S \omega_S$$

$$\frac{C_R^G}{M_T} = 2\zeta_R \omega_T$$

$$\frac{C_{\theta_R}^G}{I_T^G} = 2\zeta_{\theta_R} \omega_{\theta_T}$$

$$\frac{C_S^G}{M_T} = \frac{C_S^G}{M_S} \frac{M_S}{M_T} = 2\zeta_S \omega_S \frac{M_S}{M_T}$$

$$r_{AG}^2 M_T = \Delta I_T^G$$

$$r_{SG}^2 M_S = \Delta I_S^G$$

$$r_{RG}^2 M_T = \Delta I_R^G$$

$$\frac{r_{AG}^2 C_T^G}{I_T^G} = \frac{C_T^G}{M_T} \frac{r_{AG}^2 M_T}{I_T^G} = 2\zeta_T \omega_T \frac{r_{AG}^2 M_T}{I_T^G} = 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G}$$

$$\frac{r_{SG}^2 C_S^G}{I_T^G} = \frac{C_S^G}{M_S} \frac{r_{SG}^2 M_S}{I_T^G} = 2\zeta_S \omega_S \frac{\Delta I_S^G}{I_T^G}$$

$$\frac{r_{RG}^2 C_R^G}{I_T^G} = \frac{C_R^G}{M_T} \frac{r_{RG}^2 M_T}{I_T^G} = 2\zeta_R \omega_T \frac{r_{RG}^2 M_T}{I_T^G} = 2\zeta_R \omega_T \frac{\Delta I_R^G}{I_T^G}$$

$$\frac{K_T^G}{M_T} = \omega_T^2$$

$$\frac{K_{\theta_T}^G}{I_T^G} = \omega_{\theta_T}^2$$

$$\frac{K_S^G}{M_S} = \omega_S^2$$

$$\frac{K_R^G}{M_T} = \omega_R^2$$

$$\frac{K_{\theta_R}^G}{I_T^G} = \omega_{\theta_R}^2$$

$$\frac{K_S^G}{M_T} = \frac{K_S^G}{M_S} \frac{M_S}{M_T} = \omega_S^2 \frac{M_S}{M_T}$$

$$\frac{r_{AG}^2 K_T^G}{I_T^G} = \frac{K_T^G}{M_T} \frac{r_{AG}^2 M_T}{I_T^G} = \omega_T^2 \frac{\Delta I_T^G}{I_T^G}$$

$$\frac{r_{SG}^2 K_S^G}{I_T^G} = \frac{K_S^G}{M_S} \frac{r_{SG}^2 M_S}{I_T^G} = \omega_S^2 \frac{\Delta I_S^G}{I_T^G}$$

$$\frac{r_{RG}^2 K_R^G}{I_T^G} = \frac{K_R^G}{M_T} \frac{r_{RG}^2 M_T}{I_T^G} = \omega_R^2 \frac{\Delta I_R^G}{I_T^G}$$

The analysis will start with the equations of motion at the CG, then transform them to the table origin. The equations of motion at the CG are as follows:

Equations of motion at the CG

$$M_T s^2 X_T^G(s) = F_T(s) - F_S(s) - F_R(s)$$

$$I_T^G s^2 \theta_T(s) = T_T^G(s) - T_S^G(s) - T_R^G(s)$$

$$M_S s^2 X_S^G(s) = F_S(s)$$

where

$$F_T(s) = [C_T^G s + K_T^G] \cdot [X_A^G(s) - X_T^G(s) - r_{AG}(\theta_A(s) - \theta_T(s))]$$

$$F_S(s) = [C_S^G s + K_S^G] \cdot [X_T^G(s) - r_{SG}\theta_T(s) - X_S^G(s)]$$

$$F_R(s) = [C_R^G s + K_R^G] \cdot [X_T^G(s) - r_{RG}\theta_T(s)]$$

$$T_T^G(s) = [C_{\theta_T}^G s + K_{\theta_T}^G] \cdot [\theta_A(s) - \theta_T(s)] - r_{AG} [C_T^G s + K_T^G] \cdot [X_A^G(s) - X_T^G(s)]$$

$$T_S^G(s) = r_{GS} F_S(s)$$

$$T_R^G(s) = [C_{\theta_R}^G s + K_{\theta_R}^G] \theta_T(s) - r_{RG} [C_R^G s + K_R^G] X_T^G(s)$$

Note: The $\theta_A(s)$ term in the expression for $F_T(s)$ is not due to physical coupling, but rather due to coupling inside the controller jacobian. See the section titled *Physical System Force Equations* at the end of this document for a derivation of the force equation for $F_T(s)$ (and also $T_T^G(s)$) above.

Motion at the CG

Using the *Equations of motion at the CG*, substitute the force equations into the motion equations:

Table translational:

$$M_T s^2 X_T^G(s) = F_T(s) - F_S(s) - F_R(s)$$

$$\Rightarrow M_T s^2 X_T^G(s) = [C_T^G s + K_T^G] \cdot [X_A^G(s) - X_T^G(s) - r_{AG}(\theta_A(s) - \theta_T(s))] - [C_S^G s + K_S^G] \cdot [X_T^G(s) - r_{SG}\theta_T(s) - X_S^G(s)] - [C_R^G s + K_R^G] \cdot [X_T^G(s) - r_{RG}\theta_T(s)]$$

$$\Rightarrow M_T s^2 X_T^G(s) = [C_T^G s + K_T^G] X_A^G(s) - [C_T^G s + K_T^G] X_T^G(s) - r_{AG} [C_T^G s + K_T^G] \theta_A(s) + r_{AG} [C_T^G s + K_T^G] \theta_T(s) - [C_S^G s + K_S^G] X_T^G(s) + r_{SG} [C_S^G s + K_S^G] \theta_T(s) + [C_S^G s + K_S^G] X_S^G(s) - [C_R^G s + K_R^G] X_T^G(s) + r_{RG} [C_R^G s + K_R^G] \theta_T(s)$$

$$\Rightarrow [M_T s^2 + (C_T^G + C_S^G + C_R^G)s + (K_T^G + K_S^G + K_R^G)] X_T^G(s) = [C_T^G s + K_T^G] X_A^G(s) - r_{AG} [C_T^G s + K_T^G] \theta_A(s) + [(r_{AG} C_T^G + r_{SG} C_S^G + r_{RG} C_R^G)s + (r_{AG} K_T^G + r_{SG} K_S^G + r_{RG} K_R^G)] \theta_T(s) + [C_S^G s + K_S^G] X_S^G(s)$$

$$\begin{aligned}
& \left[M_T s^2 + (C_T^G + C_S^G + C_R^G) s + (K_T^G + K_S^G + K_R^G) \right] X_T^G(s) \\
& - \left[(r_{AG} C_T^G + r_{SG} C_S^G + r_{RG} C_R^G) s + (r_{AG} K_T^G + r_{SG} K_S^G + r_{RG} K_R^G) \right] \theta_T(s) \\
\Rightarrow & - \left[C_S^G s + K_S^G \right] X_S^G(s) \\
& = \left[C_T^G s + K_T^G \right] X_A^G(s) - r_{AG} \left[C_T^G s + K_T^G \right] \theta_A(s)
\end{aligned}$$

$$\begin{aligned}
& \left(s^2 + \frac{C_T^G + C_S^G + C_R^G}{M_T} s + \frac{K_T^G + K_S^G + K_R^G}{M_T} \right) X_T^G(s) \\
& - \left(\frac{r_{AG} C_T^G + r_{SG} C_S^G + r_{RG} C_R^G}{M_T} s + \frac{r_{AG} K_T^G + r_{SG} K_S^G + r_{RG} K_R^G}{M_T} \right) \theta_T(s) \\
\Rightarrow & - \left(\frac{C_S^G}{M_T} s + \frac{K_S^G}{M_T} \right) X_S^G(s) \\
& = \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right) X_A^G(s) - r_{AG} \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right) \theta_A(s)
\end{aligned}$$

Table rotational:

$$I_T^G s^2 \theta_T(s) = T_T^G(s) - T_S^G(s) - T_R^G(s)$$

$$\begin{aligned}
I_T^G s^2 \theta_T(s) &= \left[C_{\theta_T}^G s + K_{\theta_T}^G \right] \cdot [\theta_A(s) - \theta_T(s)] - r_{AG} \left[C_T^G s + K_T^G \right] \cdot [X_A^G(s) - X_T^G(s)] \\
\Rightarrow & - r_{GS} \left[C_S^G s + K_S^G \right] \cdot [X_T^G(s) - r_{SG} \theta_T(s) - X_S^G(s)] \\
& - \left[C_{\theta_R}^G s + K_{\theta_R}^G \right] \theta_T(s) + r_{RG} \left[C_R^G s + K_R^G \right] X_T^G(s)
\end{aligned}$$

$$\begin{aligned}
I_T^G s^2 \theta_T(s) &= \left[C_{\theta_T}^G s + K_{\theta_T}^G \right] \theta_A(s) - \left[C_{\theta_T}^G s + K_{\theta_T}^G \right] \theta_T(s) - r_{AG} \left[C_T^G s + K_T^G \right] X_A^G(s) + r_{AG} \left[C_T^G s + K_T^G \right] X_T^G(s) \\
\Rightarrow & - r_{GS} \left[C_S^G s + K_S^G \right] X_T^G(s) + r_{SG} r_{GS} \left[C_S^G s + K_S^G \right] \theta_T(s) + r_{GS} \left[C_S^G s + K_S^G \right] X_S^G(s) \\
& - \left[C_{\theta_R}^G s + K_{\theta_R}^G \right] \theta_T(s) + r_{RG} \left[C_R^G s + K_R^G \right] X_T^G(s)
\end{aligned}$$

$$\begin{aligned}
& - r_{AG} \left[C_T^G s + K_T^G \right] X_T^G(s) - r_{SG} \left[C_S^G s + K_S^G \right] X_T^G(s) - r_{RG} \left[C_R^G s + K_R^G \right] X_T^G(s) \\
I_T^G s^2 \theta_T(s) &+ \left[C_{\theta_T}^G s + K_{\theta_T}^G \right] \theta_T(s) + r_{SG}^2 \left[C_S^G s + K_S^G \right] \theta_T(s) + \left[C_{\theta_R}^G s + K_{\theta_R}^G \right] \theta_T(s) \\
\Rightarrow & + r_{SG} \left[C_S^G s + K_S^G \right] X_S^G(s) \quad (r_{GS} = -r_{SG}) \\
& = -r_{AG} \left[C_T^G s + K_T^G \right] X_A^G(s) \\
& + \left[C_{\theta_T}^G s + K_{\theta_T}^G \right] \theta_A(s)
\end{aligned}$$

$$\begin{aligned}
& - \left[(r_{AG} C_T^G + r_{SG} C_S^G + r_{RG} C_R^G) s + (r_{AG} K_T^G + r_{SG} K_S^G + r_{RG} K_R^G) \right] X_T^G(s) \\
& \left[I_T^G s^2 + (C_{\theta_T}^G + r_{SG}^2 C_S^G + C_{\theta_R}^G) s + (K_{\theta_T}^G + r_{SG}^2 K_S^G + K_{\theta_R}^G) \right] \theta_T(s) \\
\Rightarrow & + r_{SG} \left[C_S^G s + K_S^G \right] X_S^G(s) \\
& = -r_{AG} \left[C_T^G s + K_T^G \right] X_A^G(s) \\
& + \left[C_{\theta_T}^G s + K_{\theta_T}^G \right] \theta_A(s)
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{r_{AG}C_T^G + r_{SG}C_S^G + r_{RG}C_R^G}{I_T^G} s + \frac{r_{AG}K_T^G + r_{SG}K_S^G + r_{RG}K_R^G}{I_T^G} \right) X_T^G(s) \\
& \left(s^2 + \frac{C_{\theta_T}^G + r_{SG}^2 C_S^G + C_{\theta_R}^G}{I_T^G} s + \frac{K_{\theta_T}^G + r_{SG}^2 K_S^G + K_{\theta_R}^G}{I_T^G} \right) \theta_T(s) \\
\Rightarrow & + r_{SG} \left(\frac{C_S^G}{I_T^G} s + \frac{K_S^G}{I_T^G} \right) X_S^G(s) \\
& = -r_{AG} \left(\frac{C_T^G}{I_T^G} s + \frac{K_T^G}{I_T^G} \right) X_A^G(s) \\
& + \left(\frac{C_{\theta_T}^G}{I_T^G} s + \frac{K_{\theta_T}^G}{I_T^G} \right) \theta_A(s)
\end{aligned}$$

Specimen translational:

$$\begin{aligned}
& M_s s^2 X_s^G(s) = F_s(s) \\
\Rightarrow & M_s s^2 X_s^G(s) = [C_s^G s + K_s^G] \cdot [X_T^G(s) - r_{SG} \theta_T(s) - X_S^G(s)] \\
\Rightarrow & M_s s^2 X_s^G(s) = [C_s^G s + K_s^G] X_T^G(s) - r_{SG} [C_s^G s + K_s^G] \theta_T(s) - [C_s^G s + K_s^G] X_S^G(s) \\
\Rightarrow & [M_s s^2 + C_s^G s + K_s^G] X_s^G(s) = [C_s^G s + K_s^G] X_T^G(s) - r_{SG} [C_s^G s + K_s^G] \theta_T(s) \\
\Rightarrow & -[C_s^G s + K_s^G] X_T^G(s) + r_{SG} [C_s^G s + K_s^G] \theta_T(s) + [M_s s^2 + C_s^G s + K_s^G] X_s^G(s) = 0 \\
\Rightarrow & - \left(\frac{C_s^G}{M_s} s + \frac{K_s^G}{M_s} \right) X_T^G(s) + r_{SG} \left(\frac{C_s^G}{M_s} s + \frac{K_s^G}{M_s} \right) \theta_T(s) + \left(s^2 + \frac{C_s^G}{M_s} s + \frac{K_s^G}{M_s} \right) X_s^G(s) = 0
\end{aligned}$$

Motion matrix equation at the CG (in terms of physical parameters)

$$\mathbf{A}_V^G \cdot \begin{bmatrix} sX_T^G(s) \\ s\theta_T(s) \\ sX_S^G(s) \end{bmatrix} = \mathbf{B}_V^G \cdot \begin{bmatrix} sX_A^G(s) \\ s\theta_A(s) \end{bmatrix}, \quad \text{where} \quad \mathbf{A}_V^G = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{B}_V^G = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}:$$

$$a_{11} = s^2 + \frac{C_T^G + C_S^G + C_R^G}{M_T} s + \frac{K_T^G + K_S^G + K_R^G}{M_T}$$

$$a_{12} = -\left(\frac{r_{AG}C_T^G + r_{SG}C_S^G + r_{RG}C_R^G}{M_T} s + \frac{r_{AG}K_T^G + r_{SG}K_S^G + r_{RG}K_R^G}{M_T} \right)$$

$$a_{13} = -\left(\frac{C_S^G}{M_T} s + \frac{K_S^G}{M_T} \right)$$

$$a_{21} = -\left(\frac{r_{AG}C_T^G + r_{SG}C_S^G + r_{RG}C_R^G}{I_T^G} s + \frac{r_{AG}K_T^G + r_{SG}K_S^G + r_{RG}K_R^G}{I_T^G} \right)$$

$$a_{22} = s^2 + \frac{C_{\theta_T}^G + r_{SG}^2 C_S^G + C_{\theta_R}^G}{I_T^G} s + \frac{K_{\theta_T}^G + r_{SG}^2 K_S^G + K_{\theta_R}^G}{I_T^G}$$

$$a_{23} = r_{SG} \left(\frac{C_S^G}{I_T^G} s + \frac{K_S^G}{I_T^G} \right)$$

$$a_{31} = -\left(\frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S} \right)$$

$$a_{32} = r_{SG} \left(\frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S} \right)$$

$$a_{33} = s^2 + \frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S}$$

$$b_{11} = \frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T}$$

$$b_{12} = -r_{AG} \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right)$$

$$b_{21} = -r_{AG} \left(\frac{C_T^G}{I_T^G} s + \frac{K_T^G}{I_T^G} \right)$$

$$b_{22} = \frac{C_{\theta_T}^G}{I_T^G} s + \frac{K_{\theta_T}^G}{I_T^G}$$

$$b_{31} = b_{32} = 0$$

Motion matrix equation at the CG (in terms of physical parameters) (continued)

Solution of this matrix equation will yield transfer functions

$$(\mathbf{A}_V^G)^{-1} \cdot \mathbf{B}_V^G = \begin{bmatrix} \frac{sX_T^G}{sX_A^G}(s) & \frac{sX_T^G}{s\theta_A}(s) \\ \frac{s\theta_T}{sX_A^G}(s) & \frac{s\theta_T}{s\theta_A}(s) \\ \frac{sX_S^G}{sX_A^G}(s) & \frac{sX_S^G}{s\theta_A}(s) \end{bmatrix}$$

that define motion at the CG

$$sX_T^G(s) = \frac{sX_T^G}{sX_A^G}(s)sX_A^G(s) + \frac{sX_T^G}{s\theta_A}(s)s\theta_A(s)$$

$$s\theta_T(s) = \frac{s\theta_T}{sX_A^G}(s)sX_A^G(s) + \frac{s\theta_T}{s\theta_A}(s)s\theta_A(s)$$

$$sX_S^G(s) = \frac{sX_S^G}{sX_A^G}(s)sX_A^G(s) + \frac{sX_S^G}{s\theta_A}(s)s\theta_A(s)$$

The motion at the table origin can be derived by applying the following transforms to the motion at the CG:

Table origin to CG transforms

$$X_T^G(s) = X_T^T(s) + r_{TG}\theta_T(s)$$

$$T_T^G(s) = T_T^T(s) - r_{TG}F_T(s)$$

$$X_A^G(s) = X_A^T(s) + r_{TG}\theta_A(s)$$

$$T_S^G(s) = T_S^T(s) - r_{TG}F_S(s)$$

$$X_S^G(s) = X_S^T(s)$$

$$T_R^G(s) = T_R^T(s) - r_{TG}F_R(s)$$

Rewrite the above equations in matrix format:

$$\begin{bmatrix} X_T^G(s) \\ \theta_T(s) \\ X_S^G(s) \end{bmatrix} = \begin{bmatrix} 1 & r_{TG} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_T^T(s) \\ \theta_T(s) \\ X_S^T(s) \end{bmatrix}, \quad \begin{bmatrix} X_A^G(s) \\ \theta_A(s) \end{bmatrix} = \begin{bmatrix} 1 & r_{TG} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_A^T(s) \\ \theta_A(s) \end{bmatrix}$$

$$\mathbf{A}_V^G \cdot \begin{bmatrix} X_T^G(s) \\ \theta_T(s) \\ X_S^G(s) \end{bmatrix} = \mathbf{B}_V^G \cdot \begin{bmatrix} X_A^G(s) \\ \theta_A(s) \end{bmatrix} \Rightarrow \mathbf{A}_V^G \cdot \begin{bmatrix} 1 & r_{TG} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_T^T(s) \\ \theta_T(s) \\ X_S^T(s) \end{bmatrix} = \mathbf{B}_V^G \cdot \begin{bmatrix} 1 & r_{TG} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_A^T(s) \\ \theta_A(s) \end{bmatrix}$$

$$\mathbf{A}_V^T = \mathbf{A}_V^G \cdot \begin{bmatrix} 1 & r_{TG} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} 1 & r_{TG} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} + r_{TG}a_{11} & a_{13} \\ a_{21} & a_{22} + r_{TG}a_{21} & a_{23} \\ a_{31} & a_{32} + r_{TG}a_{31} & a_{33} \end{bmatrix}$$

$$a'_{12} = a_{12} + r_{TG}a_{11}$$

$$\Rightarrow a'_{12} = - \left(\frac{r_{AG}C_T^G + r_{SG}C_S^G + r_{RG}C_R^G}{M_T} s + \frac{r_{AG}K_T^G + r_{SG}K_S^G + r_{RG}K_R^G}{M_T} \right) + r_{TG} \left(s^2 + \frac{C_T^G + C_S^G + C_R^G}{M_T} s + \frac{K_T^G + K_S^G + K_R^G}{M_T} \right)$$

$$\Rightarrow a'_{12} = r_{TG}s^2 + \frac{(r_{TG} - r_{AG})C_T^G + (r_{TG} - r_{SG})C_S^G + (r_{TG} - r_{RG})C_R^G}{M_T} s + \frac{(r_{TG} - r_{AG})K_T^G + (r_{TG} - r_{SG})K_S^G + (r_{TG} - r_{RG})K_R^G}{M_T}$$

$$\Rightarrow a'_{12} = r_{TG}s^2 + \frac{(r_{TG} + r_{GA})C_T^G + (r_{TG} + r_{GS})C_S^G + (r_{TG} + r_{GR})C_R^G}{M_T} s + \frac{(r_{TG} + r_{GA})K_T^G + (r_{TG} + r_{GS})K_S^G + (r_{TG} + r_{GR})K_R^G}{M_T}$$

$$\Rightarrow a'_{12} = r_{TG}s^2 + \frac{r_{TA}C_T^G + r_{TS}C_S^G + r_{TR}C_R^G}{M_T} s + \frac{r_{TA}K_T^G + r_{TS}K_S^G + r_{TR}K_R^G}{M_T}$$

$$\Rightarrow a'_{12} = - \left(r_{GT}s^2 + \frac{r_{AT}C_T^G + r_{ST}C_S^G + r_{RT}C_R^G}{M_T} s + \frac{r_{AT}K_T^G + r_{ST}K_S^G + r_{RT}K_R^G}{M_T} \right)$$

$$a'_{22} = a_{22} + r_{TG}a_{21}$$

$$\Rightarrow a'_{22} = s^2 + \frac{C_{\theta_T}^G + r_{SG}^2 C_S^G + C_{\theta_R}^G}{I_T^G} s + \frac{K_{\theta_T}^G + r_{SG}^2 K_S^G + K_{\theta_R}^G}{I_T^G} - r_{TG} \left(\frac{r_{AG}C_T^G + r_{SG}C_S^G + r_{RG}C_R^G}{I_T^G} s + \frac{r_{AG}K_T^G + r_{SG}K_S^G + r_{RG}K_R^G}{I_T^G} \right)$$

$$\Rightarrow a'_{22} = s^2 + \frac{C_{\theta_T}^G + r_{SG}^2 C_S^G + C_{\theta_R}^G - r_{TG}(r_{AG}C_T^G + r_{SG}C_S^G + r_{RG}C_R^G)}{I_T^G} s + \frac{K_{\theta_T}^G + r_{SG}^2 K_S^G + K_{\theta_R}^G - r_{TG}(r_{AG}K_T^G + r_{SG}K_S^G + r_{RG}K_R^G)}{I_T^G}$$

$$a'_{32} = a_{32} + r_{TG}a_{31}$$

$$\Rightarrow a'_{32} = r_{SG} \left(\frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S} \right) - r_{TG} \left(\frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S} \right)$$

$$\Rightarrow a'_{32} = (r_{SG} - r_{TG}) \left(\frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S} \right)$$

$$\Rightarrow a'_{32} = (r_{SG} + r_{GT}) \left(\frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S} \right)$$

$$\Rightarrow a'_{32} = r_{ST} \left(\frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S} \right)$$

$$\mathbf{B}_V^T = \mathbf{B}_V^G \cdot \begin{bmatrix} 1 & r_{TG} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \cdot \begin{bmatrix} 1 & r_{TG} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} + r_{TG}b_{11} \\ b_{21} & b_{22} + r_{TG}b_{21} \\ b_{31} & b_{32} + r_{TG}b_{31} \end{bmatrix}$$

$$b'_{12} = b_{12} + r_{TG}b_{11}$$

$$\Rightarrow b'_{12} = -r_{AG} \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right) + r_{TG} \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right)$$

$$\Rightarrow b'_{12} = -(r_{AG} - r_{TG}) \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right)$$

$$\Rightarrow b'_{12} = -(r_{AG} + r_{GT}) \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right)$$

$$\Rightarrow b'_{12} = -r_{AT} \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right)$$

$$b'_{22} = b_{22} + r_{TG}b_{21}$$

$$\Rightarrow b'_{22} = \frac{C_{\theta_T}^G}{I_T^G} s + \frac{K_{\theta_T}^G}{I_T^G} - r_{TG}r_{AG} \left(\frac{C_T^G}{I_T^G} s + \frac{K_T^G}{I_T^G} \right)$$

$$\Rightarrow b'_{22} = \frac{C_{\theta_T}^G - r_{TG}r_{AG}C_T^G}{I_T^G} s + \frac{K_{\theta_T}^G - r_{TG}r_{AG}K_T^G}{I_T^G}$$

$$b'_{32} = b_{32} + r_{TG}b_{31} = b_{32} + r_{TG} \cdot 0 = b_{32} \quad (\text{unchanged})$$

Motion matrix equation at the table origin (in terms of physical parameters)

$$\mathbf{A}_V^T \cdot \begin{bmatrix} sX_T^T(s) \\ s\theta_T(s) \\ sX_S^T(s) \end{bmatrix} = \mathbf{B}_V^T \cdot \begin{bmatrix} sX_A^T(s) \\ s\theta_A(s) \end{bmatrix}, \quad \text{where} \quad \mathbf{A}_V^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{B}_V^T = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}:$$

$$a_{11} = s^2 + \frac{C_T^G + C_S^G + C_R^G}{M_T} s + \frac{K_T^G + K_S^G + K_R^G}{M_T}$$

$$a_{12} = - \left(r_{GT} s^2 + \frac{r_{AT} C_T^G + r_{ST} C_S^G + r_{RT} C_R^G}{M_T} s + \frac{r_{AT} K_T^G + r_{ST} K_S^G + r_{RT} K_R^G}{M_T} \right)$$

$$a_{13} = - \left(\frac{C_S^G}{M_T} s + \frac{K_S^G}{M_T} \right)$$

$$a_{21} = - \left(\frac{r_{AG} C_T^G + r_{SG} C_S^G + r_{RG} C_R^G}{I_T^G} s + \frac{r_{AG} K_T^G + r_{SG} K_S^G + r_{RG} K_R^G}{I_T^G} \right)$$

$$a_{22} = s^2 + \frac{C_{\theta_T}^G + r_{SG}^2 C_S^G + C_{\theta_R}^G - r_{TG} (r_{AG} C_T^G + r_{SG} C_S^G + \mathbf{0} \cdot r_{RG} C_R^G)}{I_T^G} s$$

$$+ \frac{K_{\theta_T}^G + r_{SG}^2 K_S^G + K_{\theta_R}^G - r_{TG} (r_{AG} K_T^G + r_{SG} K_S^G + \mathbf{0} \cdot r_{RG} K_R^G)}{I_T^G}$$

(see note below)

$$a_{23} = r_{SG} \left(\frac{C_S^G}{I_T^G} s + \frac{K_S^G}{I_T^G} \right)$$

$$a_{31} = - \left(\frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S} \right)$$

$$a_{32} = r_{ST} \left(\frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S} \right)$$

$$a_{33} = s^2 + \frac{C_S^G}{M_S} s + \frac{K_S^G}{M_S}$$

$$b_{11} = \frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T}$$

$$b_{12} = -r_{AT} \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right)$$

$$b_{21} = -r_{AG} \left(\frac{C_T^G}{I_T^G} s + \frac{K_T^G}{I_T^G} \right)$$

$$b_{22} = \frac{C_{\theta_T}^G - r_{TG} r_{AG} C_T^G}{I_T^G} s + \frac{K_{\theta_T}^G - r_{TG} r_{AG} K_T^G}{I_T^G}$$

$$b_{31} = b_{32} = 0$$

IMPORTANT NOTE: For reasons unknown, above equations are valid only when the specimen spring/damper is connected to the table origin, i.e. $r_{RG} = r_{TG}$.

Rewrite the motion matrix equation in terms of natural frequency and damping ratio:

$$\begin{aligned}
a_{11} &= s^2 + \left(\frac{C_T^G}{M_T} + \frac{C_S^G}{M_T} + \frac{C_R^G}{M_T} \right) s + \left(\frac{K_T^G}{M_T} + \frac{K_S^G}{M_T} + \frac{K_R^G}{M_T} \right) \\
&= s^2 + \left(2\zeta_T \omega_T + 2\zeta_S \omega_S \frac{M_S}{M_T} + 2\zeta_R \omega_T \right) s + \left(\omega_T^2 + \omega_S^2 \frac{M_S}{M_T} + \omega_R^2 \right) \\
a_{12} &= - \left(r_{GT} s^2 + \left(r_{AT} \frac{C_T^G}{M_T} + r_{ST} \frac{C_S^G}{M_T} + r_{RT} \frac{C_R^G}{M_T} \right) s + \left(r_{AT} \frac{K_T^G}{M_T} + r_{ST} \frac{K_S^G}{M_T} + r_{RT} \frac{K_R^G}{M_T} \right) \right) \\
&= - \left(r_{GT} s^2 + \left(r_{AT} 2\zeta_T \omega_T + r_{ST} 2\zeta_S \omega_S \frac{M_S}{M_T} + r_{RT} 2\zeta_R \omega_T \right) s + \left(r_{AT} \omega_T^2 + r_{ST} \omega_S^2 \frac{M_S}{M_T} + r_{RT} \omega_R^2 \right) \right) \\
a_{13} &= - \left(\frac{C_S^G}{M_T} s + \frac{K_S^G}{M_T} \right) \\
&= - \frac{M_S}{M_T} (2\zeta_S \omega_S s + \omega_S^2) \\
a_{21} &= - \left(\left(r_{AG} \frac{C_T^G}{I_T^G} + r_{SG} \frac{C_S^G}{I_T^G} + r_{RG} \frac{C_R^G}{I_T^G} \right) s + \left(r_{AG} \frac{K_T^G}{I_T^G} + r_{SG} \frac{K_S^G}{I_T^G} + r_{RG} \frac{K_R^G}{I_T^G} \right) \right) \\
&= - \left(\left(\frac{1}{r_{AG}} \frac{r_{AG}^2 C_T^G}{I_T^G} + \frac{1}{r_{SG}} \frac{r_{SG}^2 C_S^G}{I_T^G} + \frac{1}{r_{RG}} \frac{r_{RG}^2 C_R^G}{I_T^G} \right) s + \left(\frac{1}{r_{AG}} \frac{r_{AG}^2 K_T^G}{I_T^G} + \frac{1}{r_{SG}} \frac{r_{SG}^2 K_S^G}{I_T^G} + \frac{1}{r_{RG}} \frac{r_{RG}^2 K_R^G}{I_T^G} \right) \right) \\
&= - \left(\left(\frac{1}{r_{AG}} 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} + \frac{1}{r_{SG}} 2\zeta_S \omega_S \frac{\Delta I_S^G}{I_T^G} + \frac{1}{r_{RG}} 2\zeta_R \omega_T \frac{\Delta I_R^G}{I_T^G} \right) s + \left(\frac{1}{r_{AG}} \omega_T^2 \frac{\Delta I_T^G}{I_T^G} + \frac{1}{r_{SG}} \omega_S^2 \frac{\Delta I_S^G}{I_T^G} + \frac{1}{r_{RG}} \omega_R^2 \frac{\Delta I_R^G}{I_T^G} \right) \right) \\
a_{22} &= s^2 + \left(\frac{C_{\theta T}^G}{I_T^G} + \frac{r_{SG}^2 C_S^G}{I_T^G} + \frac{C_{\theta R}^G}{I_T^G} - r_{TG} \left(r_{AG} \frac{C_T^G}{I_T^G} + r_{SG} \frac{C_S^G}{I_T^G} + r_{RG} \frac{C_R^G}{I_T^G} \right) \right) s \\
&\quad + \frac{K_{\theta T}^G}{I_T^G} + \frac{r_{SG}^2 K_S^G}{I_T^G} + \frac{K_{\theta R}^G}{I_T^G} - r_{TG} \left(r_{AG} \frac{K_T^G}{I_T^G} + r_{SG} \frac{K_S^G}{I_T^G} + r_{RG} \frac{K_R^G}{I_T^G} \right) \\
&= s^2 + \left(\frac{C_{\theta T}^G}{I_T^G} + \frac{r_{SG}^2 C_S^G}{I_T^G} + \frac{C_{\theta R}^G}{I_T^G} - \left(\frac{r_{TG}}{r_{AG}} \frac{r_{AG}^2 C_T^G}{I_T^G} + \frac{r_{TG}}{r_{SG}} \frac{r_{SG}^2 C_S^G}{I_T^G} + \frac{r_{TG}}{r_{RG}} \frac{r_{RG}^2 C_R^G}{I_T^G} \right) \right) s \\
&\quad + \frac{K_{\theta T}^G}{I_T^G} + \frac{r_{SG}^2 K_S^G}{I_T^G} + \frac{K_{\theta R}^G}{I_T^G} - \left(\frac{r_{TG}}{r_{AG}} \frac{r_{AG}^2 K_T^G}{I_T^G} + \frac{r_{TG}}{r_{SG}} \frac{r_{SG}^2 K_S^G}{I_T^G} + \frac{r_{TG}}{r_{RG}} \frac{r_{RG}^2 K_R^G}{I_T^G} \right) \\
&= s^2 + \left(2\zeta_{\theta T} \omega_{\theta T} + 2\zeta_S \omega_S \frac{\Delta I_S^G}{I_T^G} + 2\zeta_{\theta R} \omega_{\theta T} - \frac{r_{TG}}{r_{AG}} 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} - \frac{r_{TG}}{r_{SG}} 2\zeta_S \omega_S \frac{\Delta I_S^G}{I_T^G} - \frac{r_{TG}}{r_{RG}} 2\zeta_R \omega_T \frac{\Delta I_R^G}{I_T^G} \right) s \\
&\quad + \left(\omega_{\theta T}^2 + \omega_S^2 \frac{\Delta I_S^G}{I_T^G} + \omega_{\theta R}^2 - \frac{r_{TG}}{r_{AG}} \omega_T^2 \frac{\Delta I_T^G}{I_T^G} - \frac{r_{TG}}{r_{SG}} \omega_S^2 \frac{\Delta I_S^G}{I_T^G} - \frac{r_{TG}}{r_{RG}} \omega_R^2 \frac{\Delta I_R^G}{I_T^G} \right)
\end{aligned}$$

$$\begin{aligned}
a_{23} &= r_{SG} \left(\frac{C_S^G}{I_T^G} s + \frac{K_S^G}{I_T^G} \right) \\
&= \frac{1}{r_{SG}} \left(\frac{r_{SG}^2 C_S^G}{I_T^G} s + \frac{r_{SG}^2 K_S^G}{I_T^G} \right) \\
&= \frac{1}{r_{SG}} \left(2\zeta_s \omega_s \frac{\Delta I_S^G}{I_T^G} s + \omega_s^2 \frac{\Delta I_S^G}{I_T^G} \right) \\
&= \frac{1}{r_{SG}} \frac{\Delta I_S^G}{I_T^G} (2\zeta_s \omega_s s + \omega_s^2)
\end{aligned}$$

$$\begin{aligned}
a_{31} &= - \left(\frac{C_S^G}{M_s} s + \frac{K_S^G}{M_s} \right) \\
&= - (2\zeta_s \omega_s s + \omega_s^2)
\end{aligned}$$

$$\begin{aligned}
a_{32} &= r_{ST} \left(\frac{C_S^G}{M_s} s + \frac{K_S^G}{M_s} \right) \\
&= r_{ST} (2\zeta_s \omega_s s + \omega_s^2)
\end{aligned}$$

$$\begin{aligned}
a_{33} &= s^2 + \frac{C_S^G}{M_s} s + \frac{K_S^G}{M_s} \\
&= s^2 + 2\zeta_s \omega_s s + \omega_s^2
\end{aligned}$$

$$\begin{aligned}
b_{11} &= \frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \\
&= 2\zeta_T \omega_T s + \omega_T^2
\end{aligned}$$

$$\begin{aligned}
b_{12} &= -r_{AT} \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right) \\
&= -r_{AT} (2\zeta_T \omega_T s + \omega_T^2)
\end{aligned}$$

$$\begin{aligned}
b_{21} &= -r_{AG} \left(\frac{C_T^G}{I_T^G} s + \frac{K_T^G}{I_T^G} \right) \\
&= -\frac{1}{r_{AG}} \left(\frac{r_{AG}^2 C_T^G}{I_T^G} s + \frac{r_{AG}^2 K_T^G}{I_T^G} \right) \\
&= -\frac{1}{r_{AG}} \left(2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} s + \omega_T^2 \frac{\Delta I_T^G}{I_T^G} \right) \\
&= -\frac{1}{r_{AG}} \frac{\Delta I_T^G}{I_T^G} (2\zeta_T \omega_T s + \omega_T^2)
\end{aligned}$$

$$\begin{aligned}
b_{22} &= \left(\frac{C_{\theta_T}^G}{I_T^G} - r_{TG} r_{AG} \frac{C_T^G}{I_T^G} \right) s + \left(\frac{K_{\theta_T}^G}{I_T^G} - r_{TG} r_{AG} \frac{K_T^G}{I_T^G} \right) \\
&= \left(\frac{C_{\theta_T}^G}{I_T^G} - \frac{r_{TG}}{r_{AG}} \frac{r_{AG}^2 C_T^G}{I_T^G} \right) s + \left(\frac{K_{\theta_T}^G}{I_T^G} - \frac{r_{TG}}{r_{AG}} \frac{r_{AG}^2 K_T^G}{I_T^G} \right) \\
&= \left(2\zeta_{\theta_T} \omega_{\theta_T} - \frac{r_{TG}}{r_{AG}} 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} \right) s + \left(\omega_{\theta_T}^2 - \frac{r_{TG}}{r_{AG}} \omega_T^2 \frac{\Delta I_T^G}{I_T^G} \right)
\end{aligned}$$

$$b_{31} = b_{32} = 0$$

Motion matrix equation at the table origin (in terms of natural frequency and damping ratio)

$$\mathbf{A}_V^T \cdot \begin{bmatrix} sX_T^T(s) \\ s\theta_T(s) \\ sX_S^T(s) \end{bmatrix} = \mathbf{B}_V^T \cdot \begin{bmatrix} sX_A^T(s) \\ s\theta_A(s) \end{bmatrix}, \quad \text{where} \quad \mathbf{A}_V^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{B}_V^T = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}:$$

$$a_{11} = s^2 + \left(2\zeta_T \omega_T + 2\zeta_S \omega_S \frac{M_S}{M_T} + 2\zeta_R \omega_T \right) s + \left(\omega_T^2 + \omega_S^2 \frac{M_S}{M_T} + \omega_R^2 \right)$$

$$a_{12} = - \left(r_{GT} s^2 + \left(r_{AT} 2\zeta_T \omega_T + r_{ST} 2\zeta_S \omega_S \frac{M_S}{M_T} + r_{RT} 2\zeta_R \omega_T \right) s + \left(r_{AT} \omega_T^2 + r_{ST} \omega_S^2 \frac{M_S}{M_T} + r_{RT} \omega_R^2 \right) \right)$$

$$a_{13} = - \frac{M_S}{M_T} (2\zeta_S \omega_S s + \omega_S^2)$$

$$a_{21} = - \left(\left(\frac{1}{r_{AG}} 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} + \frac{1}{r_{SG}} 2\zeta_S \omega_S \frac{\Delta I_S^G}{I_T^G} + \frac{1}{r_{RG}} 2\zeta_R \omega_T \frac{\Delta I_R^G}{I_T^G} \right) s + \left(\frac{1}{r_{AG}} \omega_T^2 \frac{\Delta I_T^G}{I_T^G} + \frac{1}{r_{SG}} \omega_S^2 \frac{\Delta I_S^G}{I_T^G} + \frac{1}{r_{RG}} \omega_R^2 \frac{\Delta I_R^G}{I_T^G} \right) \right)$$

$$a_{22} = s^2 + \left(2\zeta_{\theta_T} \omega_{\theta_T} + 2\zeta_S \omega_S \frac{\Delta I_S^G}{I_T^G} + 2\zeta_{\theta_R} \omega_{\theta_T} - \frac{r_{TG}}{r_{AG}} 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} - \frac{r_{TG}}{r_{SG}} 2\zeta_S \omega_S \frac{\Delta I_S^G}{I_T^G} - \mathbf{0} \cdot \frac{r_{TG}}{r_{RG}} 2\zeta_R \omega_T \frac{\Delta I_R^G}{I_T^G} \right) s + \left(\omega_{\theta_T}^2 + \omega_S^2 \frac{\Delta I_S^G}{I_T^G} + \omega_{\theta_R}^2 - \frac{r_{TG}}{r_{AG}} \omega_T^2 \frac{\Delta I_T^G}{I_T^G} - \frac{r_{TG}}{r_{SG}} \omega_S^2 \frac{\Delta I_S^G}{I_T^G} - \mathbf{0} \cdot \frac{r_{TG}}{r_{RG}} \omega_R^2 \frac{\Delta I_R^G}{I_T^G} \right)$$

(see note below regarding a_{22})

$$a_{23} = \frac{1}{r_{SG}} \frac{\Delta I_S^G}{I_T^G} (2\zeta_S \omega_S s + \omega_S^2)$$

$$a_{31} = -(2\zeta_S \omega_S s + \omega_S^2)$$

$$a_{32} = r_{ST} (2\zeta_S \omega_S s + \omega_S^2)$$

$$a_{33} = s^2 + 2\zeta_S \omega_S s + \omega_S^2$$

$$b_{11} = 2\zeta_T \omega_T s + \omega_T^2$$

$$b_{12} = -r_{AT} (2\zeta_T \omega_T s + \omega_T^2)$$

$$b_{21} = - \frac{1}{r_{AG}} \frac{\Delta I_T^G}{I_T^G} (2\zeta_T \omega_T s + \omega_T^2)$$

$$b_{22} = \left(2\zeta_{\theta_T} \omega_{\theta_T} - \frac{r_{TG}}{r_{AG}} 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} \right) s + \left(\omega_{\theta_T}^2 - \frac{r_{TG}}{r_{AG}} \omega_T^2 \frac{\Delta I_T^G}{I_T^G} \right)$$

$$b_{31} = b_{32} = 0$$

$$\Delta I_T^G = r_{AG}^2 M_T \quad \Delta I_S^G = r_{SG}^2 M_S \quad \Delta I_R^G = r_{RG}^2 M_T$$

Note: If $r_{AG} = 0$, $r_{SG} = 0$, or $r_{RG} = 0$, set terms multiplied by $1/r_{AG}$, or $1/r_{SG}$, or $1/r_{RG}$ to zero, respectively.

IMPORTANT NOTE: For reasons unknown, above equations are valid only when the specimen spring/damper is connected to the table origin, i.e. $r_{RG} = r_{TG}$.

Force at the CG

Find the force transfer functions

$$\begin{aligned}\frac{F_T}{sX_A^G}(s) &= \frac{F_T}{s\theta_A}(s) \\ \frac{T_T^G}{sX_A^G}(s) &= \frac{T_T^G}{s\theta_A}(s) \\ \frac{F_S}{sX_A^G}(s) &= \frac{F_S}{s\theta_A}(s)\end{aligned}$$

that define force at the CG

$$\begin{aligned}F_T(s) &= \frac{F_T}{sX_A^G}(s)sX_A^G(s) + \frac{F_T}{s\theta_A}(s)s\theta_A(s) \\ T_T^G(s) &= \frac{T_T^G}{sX_A^G}(s)sX_A^G(s) + \frac{T_T^G}{s\theta_A}(s)s\theta_A(s) \\ F_S(s) &= \frac{F_S}{sX_A^G}(s)sX_A^G(s) + \frac{F_S}{s\theta_A}(s)s\theta_A(s)\end{aligned}$$

in terms of velocity transfer functions computed previously

$$\begin{aligned}\frac{sX_T^G}{sX_A^G}(s) &= \frac{sX_T^G}{s\theta_A}(s) \\ \frac{s\theta_T}{sX_A^G}(s) &= \frac{s\theta_T}{s\theta_A}(s) \\ \frac{sX_S^G}{sX_A^G}(s) &= \frac{sX_S^G}{s\theta_A}(s)\end{aligned}$$

by substituting the following motion equations into the force equations:

$$\begin{aligned}X_T^G(s) &= \frac{sX_T^G}{sX_A^G}(s)X_A^G(s) + \frac{sX_T^G}{s\theta_A}(s)\theta_A(s) \\ \theta_T(s) &= \frac{s\theta_T}{sX_A^G}(s)X_A^G(s) + \frac{s\theta_T}{s\theta_A}(s)\theta_A(s) \\ X_S^G(s) &= \frac{sX_S^G}{sX_A^G}(s)X_A^G(s) + \frac{sX_S^G}{s\theta_A}(s)\theta_A(s)\end{aligned}$$

Table translational:

$$\begin{aligned}F_T(s) &= [C_T^G s + K_T^G] \cdot [X_A^G(s) - X_T^G(s) - r_{AG}(\theta_A(s) - \theta_T(s))] \\ \Rightarrow F_T(s) &= [C_T^G s + K_T^G] \cdot [X_A^G(s) - X_T^G(s) - r_{AG}\theta_A(s) + r_{AG}\theta_T(s)]\end{aligned}$$

$$\begin{aligned}
\Rightarrow F_T(s) &= [C_T^G s + K_T^G] \cdot \begin{bmatrix} X_A^G(s) - \left(\frac{sX_T^G}{sX_A^G}(s) X_A^G(s) + \frac{sX_T^G}{s\theta_A}(s) \theta_A(s) \right) \\ -r_{AG} \theta_A(s) + r_{AG} \left(\frac{s\theta_T}{sX_A^G}(s) X_A^G(s) + \frac{s\theta_T}{s\theta_A}(s) \theta_A(s) \right) \end{bmatrix} \\
\Rightarrow F_T(s) &= [C_T^G s + K_T^G] \cdot \left[\left(1 - \frac{sX_T^G}{sX_A^G}(s) + r_{AG} \frac{s\theta_T}{sX_A^G}(s) \right) X_A^G(s) - \left(\frac{sX_T^G}{s\theta_A}(s) + r_{AG} \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right) \theta_A(s) \right] \\
F_T(s) &= \left[\frac{1}{s} (C_T^G s + K_T^G) \cdot \left(1 - \frac{sX_T^G}{sX_A^G}(s) + r_{AG} \frac{s\theta_T}{sX_A^G}(s) \right) \right] sX_A^G(s) \\
\Rightarrow &+ \left[-\frac{1}{s} (C_T^G s + K_T^G) \cdot \left(\frac{sX_T^G}{s\theta_A}(s) + r_{AG} \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right) \right] s\theta_A(s) \\
F_T(s) &= \left[\frac{1}{s} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \left(1 - \frac{sX_T^G}{sX_A^G}(s) + r_{AG} \frac{s\theta_T}{sX_A^G}(s) \right) \right] sX_A^G(s) \\
\Rightarrow &+ \left[-\frac{1}{s} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \left(\frac{sX_T^G}{s\theta_A}(s) + r_{AG} \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right) \right] s\theta_A(s)
\end{aligned}$$

Table rotational:

$$\begin{aligned}
T_T^G(s) &= [C_{\theta_T}^G s + K_{\theta_T}^G] \cdot [\theta_A(s) - \theta_T(s)] - r_{AG} [C_T^G s + K_T^G] \cdot [X_A^G(s) - X_T^G(s)] \\
T_T^G(s) &= [C_{\theta_T}^G s + K_{\theta_T}^G] \cdot \left[\theta_A(s) - \left(\frac{s\theta_T}{sX_A^G}(s) X_A^G(s) + \frac{s\theta_T}{s\theta_A}(s) \theta_A(s) \right) \right] \\
\Rightarrow &- r_{AG} [C_T^G s + K_T^G] \cdot \left[X_A^G(s) - \left(\frac{sX_T^G}{sX_A^G}(s) X_A^G(s) + \frac{sX_T^G}{s\theta_A}(s) \theta_A(s) \right) \right] \\
T_T^G(s) &= -\frac{1}{s} \left[r_{AG} (C_T^G s + K_T^G) \cdot \left(1 - \frac{sX_T^G}{sX_A^G}(s) \right) + (C_{\theta_T}^G s + K_{\theta_T}^G) \cdot \frac{s\theta_T}{sX_A^G}(s) \right] sX_A^G(s) \\
\Rightarrow &+ \frac{1}{s} \left[r_{AG} (C_T^G s + K_T^G) \cdot \frac{sX_T^G}{s\theta_A}(s) + (C_{\theta_T}^G s + K_{\theta_T}^G) \cdot \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right] s\theta_A(s) \\
T_T^G(s) &= -\frac{1}{s} \left[r_{AG} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \left(1 - \frac{sX_T^G}{sX_A^G}(s) \right) + I_T^G (2\zeta_{\theta_T} \omega_{\theta_T} s + \omega_{\theta_T}^2) \cdot \frac{s\theta_T}{sX_A^G}(s) \right] sX_A^G(s) \\
\Rightarrow &+ \frac{1}{s} \left[r_{AG} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \frac{sX_T^G}{s\theta_A}(s) + I_T^G (2\zeta_{\theta_T} \omega_{\theta_T} s + \omega_{\theta_T}^2) \cdot \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right] s\theta_A(s)
\end{aligned}$$

Specimen translational:

$$F_s(s) = [C_s^G s + K_s^G] \cdot [X_T^G(s) - r_{SG} \theta_T(s) - X_s^G(s)]$$

$$\Rightarrow F_s(s) = [C_s^G s + K_s^G] \cdot \begin{bmatrix} \left(\frac{sX_T^G}{sX_A^G}(s) X_A^G(s) + \frac{sX_T^G}{s\theta_A}(s) \theta_A(s) \right) \\ -r_{SG} \left(\frac{s\theta_T}{sX_A^G}(s) X_A^G(s) + \frac{s\theta_T}{s\theta_A}(s) \theta_A(s) \right) \\ - \left(\frac{sX_s^G}{sX_A^G}(s) X_A^G(s) + \frac{sX_s^G}{s\theta_A}(s) \theta_A(s) \right) \end{bmatrix}$$

$$\Rightarrow F_s(s) = [C_s^G s + K_s^G] \cdot \begin{bmatrix} \left(\frac{sX_T^G}{sX_A^G}(s) - r_{SG} \frac{s\theta_T}{sX_A^G}(s) - \frac{sX_s^G}{sX_A^G}(s) \right) X_A^G(s) \\ + \left(\frac{sX_T^G}{s\theta_A}(s) - r_{SG} \frac{s\theta_T}{s\theta_A}(s) + \frac{sX_s^G}{s\theta_A}(s) \right) \theta_A(s) \end{bmatrix}$$

$$\Rightarrow F_s(s) = \left[\frac{1}{s} (C_s^G s + K_s^G) \cdot \left(\frac{sX_T^G}{sX_A^G}(s) - r_{SG} \frac{s\theta_T}{sX_A^G}(s) - \frac{sX_s^G}{sX_A^G}(s) \right) \right] sX_A^G(s) \\ + \left[\frac{1}{s} (C_s^G s + K_s^G) \cdot \left(\frac{sX_T^G}{s\theta_A}(s) - r_{SG} \frac{s\theta_T}{s\theta_A}(s) + \frac{sX_s^G}{s\theta_A}(s) \right) \right] s\theta_A(s)$$

$$\Rightarrow F_s(s) = \left[\frac{1}{s} M_s (2\zeta_s \omega_s s + \omega_s^2) \cdot \left(\frac{sX_T^G}{sX_A^G}(s) - r_{SG} \frac{s\theta_T}{sX_A^G}(s) - \frac{sX_s^G}{sX_A^G}(s) \right) \right] sX_A^G(s) \\ + \left[\frac{1}{s} M_s (2\zeta_s \omega_s s + \omega_s^2) \cdot \left(\frac{sX_T^G}{s\theta_A}(s) - r_{SG} \frac{s\theta_T}{s\theta_A}(s) + \frac{sX_s^G}{s\theta_A}(s) \right) \right] s\theta_A(s)$$

Force matrix equation at the CG

$$\begin{bmatrix} F_T(s) \\ T_T^G(s) \\ F_S(s) \end{bmatrix} = \mathbf{F}^G \cdot \begin{bmatrix} sX_A^G(s) \\ s\theta_A(s) \end{bmatrix}, \quad \text{where} \quad \mathbf{F}^G = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix}$$

In terms of physical parameters:

$$\begin{aligned} f_{11} &= \frac{F_T}{sX_A^G}(s) = \frac{1}{s} \left(C_T^G s + K_T^G \right) \cdot \left[1 - \frac{sX_T^G}{sX_A^G}(s) + r_{AG} \frac{s\theta_T}{sX_A^G}(s) \right] \\ f_{12} &= \frac{F_T}{s\theta_A}(s) = -\frac{1}{s} \left(C_T^G s + K_T^G \right) \cdot \left[\frac{sX_T^G}{s\theta_A}(s) + r_{AG} \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right] \\ f_{21} &= \frac{T_T^G}{sX_A^G}(s) = -\frac{1}{s} \left[r_{AG} \left(C_T^G s + K_T^G \right) \cdot \left(1 - \frac{sX_T^G}{sX_A^G}(s) \right) + \left(C_{\theta_T}^G s + K_{\theta_T}^G \right) \cdot \frac{s\theta_T}{sX_A^G}(s) \right] \\ f_{22} &= \frac{T_T^G}{s\theta_A}(s) = \frac{1}{s} \left[r_{AG} \left(C_T^G s + K_T^G \right) \cdot \frac{sX_T^G}{s\theta_A}(s) + \left(C_{\theta_T}^G s + K_{\theta_T}^G \right) \cdot \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right] \\ f_{31} &= \frac{F_S}{sX_A^G}(s) = \frac{1}{s} \left(C_S^G s + K_S^G \right) \cdot \left[\frac{sX_T^G}{sX_A^G}(s) - r_{SG} \frac{s\theta_T}{sX_A^G}(s) - \frac{sX_S^G}{sX_A^G}(s) \right] \\ f_{32} &= \frac{F_S}{s\theta_A}(s) = \frac{1}{s} \left(C_S^G s + K_S^G \right) \cdot \left[\frac{sX_T^G}{s\theta_A}(s) - r_{SG} \frac{s\theta_T}{s\theta_A}(s) + \frac{sX_S^G}{s\theta_A}(s) \right] \end{aligned}$$

In terms of natural frequency and damping ratio:

$$\begin{aligned} f_{11} &= \frac{F_T}{sX_A^G}(s) = \frac{1}{s} M_T \left(2\zeta_T \omega_T s + \omega_T^2 \right) \cdot \left[1 - \frac{sX_T^G}{sX_A^G}(s) + r_{AG} \frac{s\theta_T}{sX_A^G}(s) \right] \\ f_{12} &= \frac{F_T}{s\theta_A}(s) = -\frac{1}{s} M_T \left(2\zeta_T \omega_T s + \omega_T^2 \right) \cdot \left[\frac{sX_T^G}{s\theta_A}(s) + r_{AG} \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right] \\ f_{21} &= \frac{T_T^G}{sX_A^G}(s) = -\frac{1}{s} \left[r_{AG} M_T \left(2\zeta_T \omega_T s + \omega_T^2 \right) \cdot \left(1 - \frac{sX_T^G}{sX_A^G}(s) \right) + I_T^G \left(2\zeta_{\theta_T} \omega_{\theta_T} s + \omega_{\theta_T}^2 \right) \cdot \frac{s\theta_T}{sX_A^G}(s) \right] \\ f_{22} &= \frac{T_T^G}{s\theta_A}(s) = \frac{1}{s} \left[r_{AG} M_T \left(2\zeta_T \omega_T s + \omega_T^2 \right) \cdot \frac{sX_T^G}{s\theta_A}(s) + I_T^G \left(2\zeta_{\theta_T} \omega_{\theta_T} s + \omega_{\theta_T}^2 \right) \cdot \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right] \\ f_{31} &= \frac{F_S}{sX_A^G}(s) = \frac{1}{s} M_S \left(2\zeta_S \omega_S s + \omega_S^2 \right) \cdot \left[\frac{sX_T^G}{sX_A^G}(s) - r_{SG} \frac{s\theta_T}{sX_A^G}(s) - \frac{sX_S^G}{sX_A^G}(s) \right] \\ f_{32} &= \frac{F_S}{s\theta_A}(s) = \frac{1}{s} M_S \left(2\zeta_S \omega_S s + \omega_S^2 \right) \cdot \left[\frac{sX_T^G}{s\theta_A}(s) - r_{SG} \frac{s\theta_T}{s\theta_A}(s) + \frac{sX_S^G}{s\theta_A}(s) \right] \end{aligned}$$

Force at the table origin

Find the force transfer functions

$$\frac{F_T}{sX_A^T}(s) \quad \frac{F_T}{s\theta_A}(s)$$

$$\frac{T_T^T}{sX_A^T}(s) \quad \frac{T_T^T}{s\theta_A}(s)$$

$$\frac{F_S}{sX_A^T}(s) \quad \frac{F_S}{s\theta_A}(s)$$

that define force at the table origin

$$F_T(s) = \frac{F_T}{sX_A^T}(s)sX_A^T(s) + \frac{F_T}{s\theta_A}(s)s\theta_A(s)$$

$$T_T^T(s) = \frac{T_T^T}{sX_A^T}(s)sX_A^T(s) + \frac{T_T^T}{s\theta_A}(s)s\theta_A(s)$$

$$F_S(s) = \frac{F_S}{sX_A^T}(s)sX_A^T(s) + \frac{F_S}{s\theta_A}(s)s\theta_A(s)$$

in terms of velocity transfer functions computed previously

$$\frac{sX_T^T}{sX_A^T}(s) \quad \frac{sX_T^T}{s\theta_A}(s)$$

$$\frac{s\theta_T}{sX_A^T}(s) \quad \frac{s\theta_T}{s\theta_A}(s)$$

$$\frac{sX_S^T}{sX_A^T}(s) \quad \frac{sX_S^T}{s\theta_A}(s)$$

by substituting the following motion equations into the force equations:

$$X_T^T(s) = \frac{sX_T^T}{sX_A^T}(s)X_A^T(s) + \frac{sX_T^T}{s\theta_A}(s)\theta_A(s)$$

$$\theta_T(s) = \frac{s\theta_T}{sX_A^T}(s)X_A^T(s) + \frac{s\theta_T}{s\theta_A}(s)\theta_A(s)$$

$$X_S^T(s) = \frac{sX_S^T}{sX_A^T}(s)X_A^T(s) + \frac{sX_S^T}{s\theta_A}(s)\theta_A(s)$$

Table translational:

Because the force equation at the table origin is similar in form to that at the CG, the result of the previous section on force at the CG can be used directly, with the following substitutions:

$$\begin{aligned}
 \text{Superscript "G"} &\Rightarrow \text{Superscript "T"} \quad (\text{except damping and spring constants}) \\
 r_{AG} &\Rightarrow r_{AT} \\
 F_T(s) &= \left[\frac{1}{s} (C_T^G s + K_T^G) \cdot \left(1 - \frac{sX_T^T}{sX_A^T}(s) + r_{AT} \frac{s\theta_T}{sX_A^T}(s) \right) \right] sX_A^T(s) \\
 &\quad + \left[-\frac{1}{s} (C_T^G s + K_T^G) \cdot \left(\frac{sX_T^T}{s\theta_A}(s) + r_{AT} \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right) \right] s\theta_A(s) \\
 F_T(s) &= \left[\frac{1}{s} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \left(1 - \frac{sX_T^T}{sX_A^T}(s) + r_{AT} \frac{s\theta_T}{sX_A^T}(s) \right) \right] sX_A^T(s) \\
 &\quad + \left[-\frac{1}{s} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \left(\frac{sX_T^T}{s\theta_A}(s) + r_{AT} \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right) \right] s\theta_A(s)
 \end{aligned}$$

Table rotational:

Because the force equation at the table origin is similar in form to that at the CG, the result of the previous section on force at the CG can be used directly, with the following substitutions:

$$\begin{aligned}
 \text{Superscript "G"} &\Rightarrow \text{Superscript "T"} \quad (\text{except damping and spring constants}) \\
 r_{AG} &\Rightarrow r_{AT} \\
 C_{\theta_T}^G &\Rightarrow C_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) C_T^G \\
 K_{\theta_T}^G &\Rightarrow K_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) K_T^G \\
 T_T^T(s) &= -\frac{1}{s} \left[r_{AT} (C_T^G s + K_T^G) \cdot \left(1 - \frac{sX_T^T}{sX_A^T}(s) \right) \right. \\
 &\quad \left. + \left[(C_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) C_T^G) s + (K_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) K_T^G) \right] \cdot \frac{s\theta_T}{sX_A^T}(s) \right] sX_A^T(s) \\
 &\quad + \frac{1}{s} \left[r_{AT} (C_T^G s + K_T^G) \cdot \frac{sX_T^T}{s\theta_A}(s) \right. \\
 &\quad \left. + \left[(C_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) C_T^G) s + (K_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) K_T^G) \right] \cdot \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right] s\theta_A(s)
 \end{aligned}$$

$$\begin{aligned}
& \left[\left(C_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) C_T^G \right) s + \left(K_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) K_T^G \right) \right] \\
&= I_T^G \left(\frac{C_{\theta_T}^G}{I_T^G} - \frac{r_{AG}^2 C_T^G}{I_T^G} \right) s + I_T^G \left(\frac{K_{\theta_T}^G}{I_T^G} - \frac{r_{AG}^2 K_T^G}{I_T^G} \right) + r_{AT}^2 M_T \left(\frac{C_T^G}{M_T} s + \frac{K_T^G}{M_T} \right) \\
&= I_T^G \left(2\zeta_{\theta_T} \omega_{\theta_T} - 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} \right) s + I_T^G \left(\omega_{\theta_T}^2 - \omega_T^2 \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T (2\zeta_T \omega_T s + \omega_T^2) \\
&= \left(I_T^G \left(2\zeta_{\theta_T} \omega_{\theta_T} - 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T (2\zeta_T \omega_T) \right) s + I_T^G \left(\omega_{\theta_T}^2 - \omega_T^2 \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T \omega_T^2 \\
\\
T_T^T(s) &= -\frac{1}{s} \left[\begin{aligned} & r_{AT} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \left(1 - \frac{s X_T^T}{s X_A^T}(s) \right) \\ & + \left[\left(I_T^G \left(2\zeta_{\theta_T} \omega_{\theta_T} - 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T (2\zeta_T \omega_T) \right) s \right. \\ & \quad \left. + I_T^G \left(\omega_{\theta_T}^2 - \omega_T^2 \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T \omega_T^2 \right] \cdot \frac{s \theta_T}{s X_A^T}(s) \end{aligned} \right] s X_A^T(s) \\
&+ \frac{1}{s} \left[\begin{aligned} & r_{AT} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \frac{s X_T^T}{s \theta_A}(s) \\ & + \left[\left(I_T^G \left(2\zeta_{\theta_T} \omega_{\theta_T} - 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T (2\zeta_T \omega_T) \right) s \right. \\ & \quad \left. + I_T^G \left(\omega_{\theta_T}^2 - \omega_T^2 \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T \omega_T^2 \right] \cdot \left(1 - \frac{s \theta_T}{s \theta_A}(s) \right) \end{aligned} \right] s \theta_A(s)
\end{aligned}$$

Specimen translational:

Because the force equation at the table origin is similar in form to that at the CG, the result of the previous section on force at the CG can be used directly, with the following substitutions:

$$\begin{aligned}
\text{Superscript "G"} &\Rightarrow \text{Superscript "T"} \quad (\text{except damping and spring constants}) \\
r_{SG} &\Rightarrow r_{ST}
\end{aligned}$$

$$\begin{aligned}
F_S(s) &= \left[\frac{1}{s} (C_S^G s + K_S^G) \cdot \left(\frac{s X_T^T}{s X_A^T}(s) - r_{ST} \frac{s \theta_T}{s X_A^T}(s) - \frac{s X_S^T}{s X_A^T}(s) \right) \right] s X_A^T(s) \\
&+ \left[\frac{1}{s} (C_S^G s + K_S^G) \cdot \left(\frac{s X_T^T}{s \theta_A}(s) - r_{ST} \frac{s \theta_T}{s \theta_A}(s) + \frac{s X_S^T}{s \theta_A}(s) \right) \right] s \theta_A(s) \\
\\
F_S(s) &= \left[\frac{1}{s} M_s (2\zeta_s \omega_s s + \omega_s^2) \cdot \left(\frac{s X_T^T}{s X_A^T}(s) - r_{ST} \frac{s \theta_T}{s X_A^T}(s) - \frac{s X_S^T}{s X_A^T}(s) \right) \right] s X_A^T(s) \\
&+ \left[\frac{1}{s} M_s (2\zeta_s \omega_s s + \omega_s^2) \cdot \left(\frac{s X_T^T}{s \theta_A}(s) - r_{ST} \frac{s \theta_T}{s \theta_A}(s) + \frac{s X_S^T}{s \theta_A}(s) \right) \right] s \theta_A(s)
\end{aligned}$$

Force matrix equation at the table origin

$$\begin{bmatrix} F_T(s) \\ T_T^T(s) \\ F_S(s) \end{bmatrix} = \mathbf{F}^T \cdot \begin{bmatrix} sX_A^T(s) \\ s\theta_A(s) \end{bmatrix}, \quad \text{where} \quad \mathbf{F}^T = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix}$$

In terms of physical parameters:

$$\begin{aligned} f_{11} &= \frac{F_T}{sX_A^T}(s) = \frac{1}{s} \left(C_T^G s + K_T^G \right) \cdot \left(1 - \frac{sX_T^T}{sX_A^T}(s) + r_{AT} \frac{s\theta_T}{sX_A^T}(s) \right) \\ f_{12} &= \frac{F_T}{s\theta_A}(s) = -\frac{1}{s} \left(C_T^G s + K_T^G \right) \cdot \left(\frac{sX_T^T}{s\theta_A}(s) + r_{AT} \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right) \\ f_{21} &= \frac{T_T^T}{sX_A^T}(s) = -\frac{1}{s} \left[r_{AT} \left(C_T^G s + K_T^G \right) \cdot \left(1 - \frac{sX_T^T}{sX_A^T}(s) \right) \right. \\ &\quad \left. + \left[\left(C_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) C_T^G \right) s + \left(K_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) K_T^G \right) \right] \cdot \frac{s\theta_T}{sX_A^T}(s) \right] \\ f_{22} &= \frac{T_T^T}{s\theta_A}(s) = \frac{1}{s} \left[r_{AT} \left(C_T^G s + K_T^G \right) \cdot \frac{sX_T^T}{s\theta_A}(s) \right. \\ &\quad \left. + \left[\left(C_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) C_T^G \right) s + \left(K_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) K_T^G \right) \right] \cdot \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right] \\ f_{31} &= \frac{F_S}{sX_A^T}(s) = \frac{1}{s} \left(C_S^G s + K_S^G \right) \cdot \left(\frac{sX_T^T}{sX_A^T}(s) - r_{ST} \frac{s\theta_T}{sX_A^T}(s) - \frac{sX_S^T}{sX_A^T}(s) \right) \\ f_{32} &= \frac{F_S}{s\theta_A}(s) = \frac{1}{s} \left(C_S^G s + K_S^G \right) \cdot \left(\frac{sX_T^T}{s\theta_A}(s) - r_{ST} \frac{s\theta_T}{s\theta_A}(s) + \frac{sX_S^T}{s\theta_A}(s) \right) \end{aligned}$$

In terms of natural frequency and damping ratio:

$$\begin{aligned} f_{11} &= \frac{F_T}{sX_A^T}(s) = \frac{1}{s} M_T \left(2\zeta_T \omega_T s + \omega_T^2 \right) \cdot \left(1 - \frac{sX_T^T}{sX_A^T}(s) + r_{AT} \frac{s\theta_T}{sX_A^T}(s) \right) \\ f_{12} &= \frac{F_T}{s\theta_A}(s) = -\frac{1}{s} M_T \left(2\zeta_T \omega_T s + \omega_T^2 \right) \cdot \left(\frac{sX_T^T}{s\theta_A}(s) + r_{AT} \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \right) \\ f_{21} &= \frac{T_T^T}{sX_A^T}(s) = -\frac{1}{s} \left[r_{AT} M_T \left(2\zeta_T \omega_T s + \omega_T^2 \right) \cdot \left(1 - \frac{sX_T^T}{sX_A^T}(s) \right) \right. \\ &\quad \left. + \left[\left(I_T^G \left(2\zeta_{\theta_T} \omega_{\theta_T} - 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T (2\zeta_T \omega_T) \right) s \right. \right. \\ &\quad \left. \left. + I_T^G \left(\omega_{\theta_T}^2 - \omega_T^2 \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T \omega_T^2 \right] \cdot \frac{s\theta_T}{sX_A^T}(s) \right] \end{aligned}$$

Force matrix equation at the table origin (continued)

$$f_{22} = \frac{T_T^T}{s\theta_A}(s) = \frac{1}{s} \left[\begin{aligned} & r_{AT} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \frac{sX_T^T}{s\theta_A}(s) \\ & + \left[\left(I_T^G \left(2\zeta_{\theta_T} \omega_{\theta_T} - 2\zeta_T \omega_T \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T (2\zeta_T \omega_T) \right) s \right. \\ & \quad \left. + I_T^G \left(\omega_{\theta_T}^2 - \omega_T^2 \frac{\Delta I_T^G}{I_T^G} \right) + r_{AT}^2 M_T \omega_T^2 \right] \cdot \left(1 - \frac{s\theta_T}{s\theta_A}(s) \right) \end{aligned} \right]$$

$$f_{31} = \frac{F_S}{sX_A^T}(s) = \frac{1}{s} M_S (2\zeta_S \omega_S s + \omega_S^2) \cdot \left(\frac{sX_T^T}{sX_A^T}(s) - r_{ST} \frac{s\theta_T}{sX_A^T}(s) - \frac{sX_S^T}{sX_A^T}(s) \right)$$

$$f_{32} = \frac{F_S}{s\theta_A}(s) = \frac{1}{s} M_S (2\zeta_S \omega_S s + \omega_S^2) \cdot \left(\frac{sX_T^T}{s\theta_A}(s) - r_{ST} \frac{s\theta_T}{s\theta_A}(s) + \frac{sX_S^T}{s\theta_A}(s) \right)$$

Reduced order system: Single translational DOF

Set all radius vectors ($r_{TG}, r_{TA}, r_{TS}, r_{AG}, r_{SG}, r_{RG}$) and rotational terms ($\theta_T(s), T_T(s)$) to zero:

Motion matrix equation (single translational DOF)

$$\mathbf{A}_V \cdot \begin{bmatrix} sX_T(s) \\ sX_S(s) \end{bmatrix} = \mathbf{B}_V \cdot \begin{bmatrix} sX_A(s) \\ 0 \end{bmatrix}, \quad \text{where} \quad \mathbf{A}_V = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B}_V = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}:$$

$$a_{11} = s^2 + 2 \left(\zeta_T \omega_T + \zeta_S \omega_S \frac{M_S}{M_T} + \zeta_R \omega_T \right) s + \left(\omega_T^2 + \omega_S^2 \frac{M_S}{M_T} + \omega_R^2 \right)$$

$$a_{12} = -\frac{M_S}{M_T} (2\zeta_S \omega_S s + \omega_S^2)$$

$$a_{21} = -(2\zeta_S \omega_S s + \omega_S^2)$$

$$a_{22} = s^2 + 2\zeta_S \omega_S s + \omega_S^2$$

$$b_1 = 2\zeta_T \omega_T s + \omega_T^2$$

$$b_2 = 0$$

Force matrix equation (single translational DOF)

$$\begin{bmatrix} F_T(s) \\ F_S(s) \end{bmatrix} = \mathbf{F}^G \cdot \begin{bmatrix} sX_A(s) \\ 0 \end{bmatrix}, \quad \text{where} \quad \mathbf{F}^G = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}:$$

$$f_1 = \frac{F_T}{sX_A}(s) = \frac{1}{s} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \left[1 - \frac{sX_T}{sX_A}(s) \right]$$

$$f_2 = \frac{F_S}{sX_A}(s) = \frac{1}{s} M_S (2\zeta_S \omega_S s + \omega_S^2) \cdot \left[\frac{sX_T}{sX_A}(s) - \frac{sX_S}{sX_A}(s) \right]$$

For analysis of single translation DOF systems, it is useful to solve these matrix equations for the table velocity transfer function $\frac{sX_T}{sX_A}(s)$, table force transfer function $\frac{F_T}{sX_A}(s)$, specimen velocity transfer function

$\frac{sX_S}{sX_A}(s)$, and specimen force transfer function $\frac{F_S}{sX_A}(s)$.

$$\begin{bmatrix} \frac{sX_T}{sX_A}(s) \\ \frac{sX_S}{sX_A}(s) \end{bmatrix} = \mathbf{A}_V^{-1} \mathbf{B}_V = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ 0 \end{bmatrix}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{bmatrix} a_{22} \\ -a_{21} \end{bmatrix} b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

Find the table velocity transfer function $\frac{sX_T}{sX_A}(s)$:

$$\frac{sX_T}{sX_A}(s) = \frac{a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

The numerator of $\frac{sX_T}{sX_A}(s)$ is:

$$a_{22}b_1 = (s^2 + 2\zeta_s\omega_s s + \omega_s^2)(2\zeta_T\omega_T s + \omega_T^2)$$

$$\Rightarrow (s^2 + 2\zeta_s\omega_s s + \omega_s^2) \cdot 2\zeta_T\omega_T s + (s^2 + 2\zeta_s\omega_s s + \omega_s^2) \cdot \omega_T^2$$

$$\Rightarrow 2\zeta_T\omega_T s^3 + 2\zeta_T\omega_T 2\zeta_s\omega_s s^2 + 2\zeta_T\omega_T \omega_s^2 s + \omega_T^2 s^2 + \omega_T^2 2\zeta_s\omega_s s + \omega_T^2 \omega_s^2$$

$$\Rightarrow 2\zeta_T\omega_T s^3 + (\omega_T^2 + 2\zeta_T\omega_T 2\zeta_s\omega_s)s^2 + (2\zeta_T\omega_T \omega_s^2 + 2\zeta_s\omega_s \omega_T^2)s + \omega_T^2 \omega_s^2$$

The denominator of the on-diagonal product is:

$$a_{11}a_{22} = \left\{ s^2 + \left(2\zeta_T\omega_T + 2\zeta_s\omega_s \frac{M_s}{M_T} + 2\zeta_R\omega_T \right) s + \left(\omega_T^2 + \omega_s^2 \frac{M_s}{M_T} + \omega_R^2 \right) \right\} \cdot \{ s^2 + 2\zeta_s\omega_s s + \omega_s^2 \}$$

$$\left\{ s^2 + \left(2\zeta_T\omega_T + 2\zeta_s\omega_s \frac{M_s}{M_T} + 2\zeta_R\omega_T \right) s + \left(\omega_T^2 + \omega_s^2 \frac{M_s}{M_T} + \omega_R^2 \right) \right\} \cdot s^2 +$$

$$\Rightarrow \left\{ s^2 + \left(2\zeta_T\omega_T + 2\zeta_s\omega_s \frac{M_s}{M_T} + 2\zeta_R\omega_T \right) s + \left(\omega_T^2 + \omega_s^2 \frac{M_s}{M_T} + \omega_R^2 \right) \right\} \cdot 2\zeta_s\omega_s s +$$

$$\left\{ s^2 + \left(2\zeta_T\omega_T + 2\zeta_s\omega_s \frac{M_s}{M_T} + 2\zeta_R\omega_T \right) s + \left(\omega_T^2 + \omega_s^2 \frac{M_s}{M_T} + \omega_R^2 \right) \right\} \cdot \omega_s^2$$

$$s^4 + \left(2\zeta_T\omega_T + 2\zeta_s\omega_s \frac{M_s}{M_T} + 2\zeta_R\omega_T \right) s^3 + \left(\omega_T^2 + \omega_s^2 \frac{M_s}{M_T} + \omega_R^2 \right) s^2 +$$

$$\Rightarrow 2\zeta_s\omega_s s^3 + \left(2\zeta_T\omega_T + 2\zeta_s\omega_s \frac{M_s}{M_T} + 2\zeta_R\omega_T \right) 2\zeta_s\omega_s s^2 + \left(\omega_T^2 + \omega_s^2 \frac{M_s}{M_T} + \omega_R^2 \right) 2\zeta_s\omega_s s +$$

$$\omega_s^2 s^2 + \left(2\zeta_T\omega_T + 2\zeta_s\omega_s \frac{M_s}{M_T} + 2\zeta_R\omega_T \right) \omega_s^2 s + \left(\omega_T^2 + \omega_s^2 \frac{M_s}{M_T} + \omega_R^2 \right) \omega_s^2$$

$$\begin{aligned}
& s^4 \\
& + \left(2\zeta_T \omega_T + 2\zeta_S \omega_S \frac{M_S}{M_T} + 2\zeta_R \omega_T \right) s^3 + 2\zeta_S \omega_S s^3 \\
\Rightarrow & + \left(\omega_T^2 + \omega_S^2 \frac{M_S}{M_T} + \omega_R^2 \right) s^2 + \left(2\zeta_T \omega_T + 2\zeta_S \omega_S \frac{M_S}{M_T} + 2\zeta_R \omega_T \right) 2\zeta_S \omega_S s^2 + \omega_S^2 s^2 \\
& + \left(\omega_T^2 + \omega_S^2 \frac{M_S}{M_T} + \omega_R^2 \right) 2\zeta_S \omega_S s + \left(2\zeta_T \omega_T + 2\zeta_S \omega_S \frac{M_S}{M_T} + 2\zeta_R \omega_T \right) \omega_S^2 s \\
& + \left(\omega_T^2 + \omega_S^2 \frac{M_S}{M_T} + \omega_R^2 \right) \omega_S^2
\end{aligned}$$

The denominator of the off-diagonal product is:

$$\begin{aligned}
a_{12}a_{21} &= - \left(-\frac{M_S}{M_T} \right) \left(2\zeta_S \omega_S s + \omega_S^2 \right) \cdot \left(2\zeta_S \omega_S s + \omega_S^2 \right) \\
\Rightarrow & \frac{M_S}{M_T} \left((2\zeta_S \omega_S s + \omega_S^2) 2\zeta_S \omega_S s + (2\zeta_S \omega_S s + \omega_S^2) \omega_S^2 \right) \\
\Rightarrow & \frac{M_S}{M_T} \left(2\zeta_S \omega_S 2\zeta_S \omega_S s^2 + 2\omega_S^2 2\zeta_S \omega_S s + \omega_S^2 \omega_S^2 \right)
\end{aligned}$$

Combining the denominator on- and off-diagonal terms:

$$\begin{aligned}
& s^4 \\
& + \left(2\zeta_T \omega_T + 2\zeta_S \omega_S \frac{M_S}{M_T} + 2\zeta_R \omega_T \right) s^3 + 2\zeta_S \omega_S s^3 \\
& + \left(\omega_T^2 + \omega_S^2 \frac{M_S}{M_T} + \omega_R^2 \right) s^2 + \left(2\zeta_T \omega_T + 2\zeta_S \omega_S \frac{M_S}{M_T} + 2\zeta_R \omega_T \right) 2\zeta_S \omega_S s^2 + \omega_S^2 s^2 - 2\zeta_S \omega_S \frac{M_S}{M_T} 2\zeta_S \omega_S s^2 \\
& + \left(\omega_T^2 + \omega_S^2 \frac{M_S}{M_T} + \omega_R^2 \right) 2\zeta_S \omega_S s + \left(2\zeta_T \omega_T + 2\zeta_S \omega_S \frac{M_S}{M_T} + 2\zeta_R \omega_T \right) \omega_S^2 s - 4\zeta_S \omega_S \frac{M_S}{M_T} \omega_S^2 s \\
& + \left(\omega_T^2 + \omega_S^2 \frac{M_S}{M_T} + \omega_R^2 \right) \omega_S^2 - \omega_S^2 \omega_S^2 \frac{M_S}{M_T} \\
& s^4 \\
& + \left(2\zeta_T \omega_T + 2\zeta_S \omega_S + 2\zeta_S \omega_S \frac{M_S}{M_T} + 2\zeta_R \omega_T \right) s^3 \\
\Rightarrow & + \left(\omega_T^2 + \omega_S^2 + \omega_S^2 \frac{M_S}{M_T} + \omega_R^2 \right) s^2 + (2\zeta_T \omega_T + 2\zeta_R \omega_T) 2\zeta_S \omega_S s^2 + \left(2\zeta_S \omega_S \frac{M_S}{M_T} 2\zeta_S \omega_S - 2\zeta_S \omega_S \frac{M_S}{M_T} 2\zeta_S \omega_S \right) s^2 \\
& + 2\zeta_S \omega_S (\omega_T^2 + \omega_R^2) s + (2\zeta_T \omega_T + 2\zeta_R \omega_T) \omega_S^2 s + \left(2\zeta_S \omega_S \frac{M_S}{M_T} \omega_S^2 + 2\zeta_S \omega_S \frac{M_S}{M_T} \omega_S^2 - 4\zeta_S \omega_S \frac{M_S}{M_T} \omega_S^2 \right) s \\
& + (\omega_T^2 + \omega_R^2) \omega_S^2 + \left(\omega_S^2 \omega_S^2 \frac{M_S}{M_T} - \omega_S^2 \omega_S^2 \frac{M_S}{M_T} \right)
\end{aligned}$$

$$\begin{aligned}
& s^4 \\
& + \left(2\zeta_T \omega_T + 2\zeta_S \omega_S \left(1 + \frac{M_S}{M_T} \right) + 2\zeta_R \omega_T \right) s^3 \\
\Rightarrow & \left(\omega_T^2 + \omega_S^2 \left(1 + \frac{M_S}{M_T} \right) + \omega_R^2 + (2\zeta_T \omega_T + 2\zeta_R \omega_T) 2\zeta_S \omega_S \right) s^2 \\
& + (2\zeta_S \omega_S (\omega_T^2 + \omega_R^2) + (2\zeta_T \omega_T + 2\zeta_R \omega_T) \omega_S^2) s \\
& + (\omega_T^2 + \omega_R^2) \omega_S^2
\end{aligned}$$

Defining mass ratio $\bar{M} = 1 + \frac{M_S}{M_T}$, the denominator becomes

$$\begin{aligned}
& s^4 + (2\zeta_T \omega_T + 2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T) s^3 + (\omega_T^2 + \omega_S^2 \bar{M} + \omega_R^2 + (2\zeta_T \omega_T + 2\zeta_R \omega_T) 2\zeta_S \omega_S) s^2 \\
& + (2\zeta_S \omega_S (\omega_T^2 + \omega_R^2) + (2\zeta_T \omega_T + 2\zeta_R \omega_T) \omega_S^2) s + (\omega_T^2 + \omega_R^2) \omega_S^2
\end{aligned}$$

Thus the table velocity transfer function for a single translation DOF is:

$$\frac{sX_T}{sX_A}(s) = \frac{2\zeta_T \omega_T s^3 + (\omega_T^2 + 2\zeta_T \omega_T 2\zeta_S \omega_S) s^2 + (2\zeta_T \omega_T \omega_S^2 + 2\zeta_S \omega_S \omega_T^2) s + \omega_T^2 \omega_S^2}{s^4 + (2\zeta_T \omega_T + 2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T) s^3 + (\omega_T^2 + \omega_S^2 \bar{M} + \omega_R^2 + (2\zeta_T \omega_T + 2\zeta_R \omega_T) 2\zeta_S \omega_S) s^2 + (2\zeta_S \omega_S (\omega_T^2 + \omega_R^2) + (2\zeta_T \omega_T + 2\zeta_R \omega_T) \omega_S^2) s + (\omega_T^2 + \omega_R^2) \omega_S^2}$$

Find the table force transfer function $\frac{F_T}{sX_A}(s)$:

$$\frac{F_T}{sX_A}(s) = \frac{1}{s} M_T (2\zeta_T \omega_T s + \omega_T^2) \cdot \left[1 - \frac{sX_T}{sX_A}(s) \right]$$

The numerator of $1 - \frac{sX_T}{sX_A}(s)$ is:

$$\begin{aligned}
& s^4 + (2\zeta_T \omega_T + 2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T) s^3 + (\omega_T^2 + \omega_S^2 \bar{M} + \omega_R^2 + (2\zeta_T \omega_T + 2\zeta_R \omega_T) 2\zeta_S \omega_S) s^2 \\
& + (2\zeta_S \omega_S (\omega_T^2 + \omega_R^2) + (2\zeta_T \omega_T + 2\zeta_R \omega_T) \omega_S^2) s + (\omega_T^2 + \omega_R^2) \omega_S^2 \\
& - [2\zeta_T \omega_T s^3 + (\omega_T^2 + 2\zeta_T \omega_T 2\zeta_S \omega_S) s^2 + (2\zeta_T \omega_T \omega_S^2 + 2\zeta_S \omega_S \omega_T^2) s + \omega_T^2 \omega_S^2] \\
\Rightarrow & s^4 + ((2\zeta_T \omega_T - 2\zeta_T \omega_T) + 2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T) s^3 \\
& + ((\omega_T^2 - \omega_T^2) + \omega_S^2 \bar{M} + \omega_R^2 + (2\zeta_T \omega_T 2\zeta_S \omega_S - 2\zeta_T \omega_T 2\zeta_S \omega_S) + 2\zeta_R \omega_T 2\zeta_S \omega_S) s^2 \\
& + ((2\zeta_S \omega_S \omega_T^2 - 2\zeta_S \omega_S \omega_T^2) + 2\zeta_S \omega_S \omega_R^2 + (2\zeta_T \omega_T \omega_S^2 - 2\zeta_T \omega_T \omega_S^2) + 2\zeta_R \omega_T \omega_S^2) s \\
& + (\omega_T^2 \omega_S^2 - \omega_T^2 \omega_S^2) + \omega_R^2 \omega_S^2 \\
\Rightarrow & s^4 + (2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T) s^3 + (\omega_S^2 \bar{M} + \omega_R^2 + 2\zeta_S \omega_S 2\zeta_R \omega_T) s^2 + (2\zeta_S \omega_S \omega_R^2 + 2\zeta_R \omega_T \omega_S^2) s + \omega_R^2 \omega_S^2
\end{aligned}$$

Multiply the above by $2\zeta_T \omega_T s + \omega_T^2$:

$$\begin{aligned}
& (s^4 + (2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T) s^3 + (\omega_S^2 \bar{M} + \omega_R^2 + 2\zeta_S \omega_S 2\zeta_R \omega_T) s^2 + (2\zeta_S \omega_S \omega_R^2 + 2\zeta_R \omega_T \omega_S^2) s + \omega_R^2 \omega_S^2) \cdot 2\zeta_T \omega_T s + \\
& (s^4 + (2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T) s^3 + (\omega_S^2 \bar{M} + \omega_R^2 + 2\zeta_S \omega_S 2\zeta_R \omega_T) s^2 + (2\zeta_S \omega_S \omega_R^2 + 2\zeta_R \omega_T \omega_S^2) s + \omega_R^2 \omega_S^2) \cdot \omega_T^2 \\
& 2\zeta_T \omega_T s^5 \\
& + (\omega_T^2 + 2\zeta_T \omega_T (2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T)) s^4 \\
\Rightarrow & + (2\zeta_T \omega_T (\omega_S^2 \bar{M} + \omega_R^2 + 2\zeta_S \omega_S 2\zeta_R \omega_T) + \omega_T^2 (2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T)) s^3 \\
& + (2\zeta_T \omega_T (2\zeta_S \omega_S \omega_R^2 + 2\zeta_R \omega_T \omega_S^2) + \omega_T^2 (\omega_S^2 \bar{M} + \omega_R^2 + 2\zeta_S \omega_S 2\zeta_R \omega_T)) s^2 \\
& + (2\zeta_T \omega_T \omega_R^2 \omega_S^2 + \omega_T^2 (2\zeta_S \omega_S \omega_R^2 + 2\zeta_R \omega_T \omega_S^2)) s \\
& + \omega_T^2 \omega_S^2 \omega_R^2
\end{aligned}$$

The denominator of $1 - \frac{sX_T}{sX_A}(s)$ is the same as that of $\frac{sX_T}{sX_A}(s)$.

Thus the table force transfer function for a single translation DOF is:

$$\begin{aligned}
\frac{F_T}{sX_A}(s) &= \frac{1}{s} M_T (2\zeta_T \omega_T s + \omega_T^2) \\
&\cdot \frac{s^4 + (2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T) s^3 + (\omega_S^2 \bar{M} + \omega_R^2 + 2\zeta_S \omega_S 2\zeta_R \omega_T) s^2 + (2\zeta_S \omega_S \omega_R^2 + 2\zeta_R \omega_T \omega_S^2) s + \omega_R^2 \omega_S^2}{s^4 + (2\zeta_T \omega_T + 2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T) s^3 + (\omega_T^2 + \omega_S^2 \bar{M} + \omega_R^2 + (2\zeta_T \omega_T + 2\zeta_R \omega_T) 2\zeta_S \omega_S) s^2 \\
&+ (2\zeta_S \omega_S (\omega_T^2 + \omega_R^2) + (2\zeta_T \omega_T + 2\zeta_R \omega_T) \omega_S^2) s + (\omega_T^2 + \omega_R^2) \omega_S^2}
\end{aligned}$$

Find the specimen velocity transfer function $\frac{sX_S}{sX_A}(s)$:

The numerator of $\frac{sX_S}{sX_A}(s)$ is:

$$\begin{aligned}
& -a_{21} b_1 = (2\zeta_S \omega_S s + \omega_S^2) \cdot (2\zeta_T \omega_T s + \omega_T^2) \\
\Rightarrow & (2\zeta_S \omega_S s + \omega_S^2) \cdot 2\zeta_T \omega_T s + \\
& (2\zeta_S \omega_S s + \omega_S^2) \cdot \omega_T^2 \\
\Rightarrow & 2\zeta_T \omega_T 2\zeta_S \omega_S s^2 + (2\zeta_T \omega_T \omega_S^2 + 2\zeta_S \omega_S \omega_T^2) s + \omega_T^2 \omega_S^2
\end{aligned}$$

Thus the specimen velocity transfer function for a single translation DOF is:

$$\begin{aligned}
\frac{sX_S}{sX_A}(s) &= \frac{2\zeta_T \omega_T 2\zeta_S \omega_S s^2 + (2\zeta_T \omega_T \omega_S^2 + 2\zeta_S \omega_S \omega_T^2) s + \omega_T^2 \omega_S^2}{s^4 + (2\zeta_T \omega_T + 2\zeta_S \omega_S \bar{M} + 2\zeta_R \omega_T) s^3 + (\omega_T^2 + \omega_S^2 \bar{M} + \omega_R^2 + (2\zeta_T \omega_T + 2\zeta_R \omega_T) 2\zeta_S \omega_S) s^2 \\
&+ (2\zeta_S \omega_S (\omega_T^2 + \omega_R^2) + (2\zeta_T \omega_T + 2\zeta_R \omega_T) \omega_S^2) s + (\omega_T^2 + \omega_R^2) \omega_S^2}
\end{aligned}$$

Find the specimen force transfer function $\frac{F_s}{sX_A}(s)$:

$$\frac{F_s}{sX_A}(s) = \frac{1}{s} M_s [2\zeta_s \omega_s s + \omega_s^2] \cdot \left[\frac{sX_T}{sX_A}(s) - \frac{sX_S}{sX_A}(s) \right]$$

$$\Rightarrow \frac{F_s}{sX_A}(s) = \frac{1}{s} M_s [2\zeta_s \omega_s s + \omega_s^2] \cdot \frac{a_{22}b_1 + a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}(s)$$

The numerator of $\frac{F_s}{sX_A}(s)$ is:

$$\begin{aligned} & M_s (2\zeta_s \omega_s s + \omega_s^2) \cdot (a_{22}b_1 + a_{21}b_1) \\ \Rightarrow & M_s (2\zeta_s \omega_s s + \omega_s^2) \cdot (s^2 + 2\zeta_s \omega_s s + \omega_s^2 - (2\zeta_s \omega_s s + \omega_s^2)) \cdot (2\zeta_T \omega_T s + \omega_T^2) \\ \Rightarrow & s^2 M_s (2\zeta_s \omega_s s + \omega_s^2) \cdot (2\zeta_T \omega_T s + \omega_T^2) \\ \Rightarrow & s^2 M_s (2\zeta_T \omega_T 2\zeta_s \omega_s s^2 + (2\zeta_T \omega_T \omega_s^2 + 2\zeta_s \omega_s \omega_T^2)s + \omega_T^2 \omega_s^2) \end{aligned}$$

Thus the specimen force transfer function for a single translation DOF is:

$$\frac{F_s}{sX_A}(s) = \frac{1}{s} \cdot \frac{s^2 M_s (2\zeta_s \omega_s s + \omega_s^2) \cdot (2\zeta_T \omega_T s + \omega_T^2)}{s^4 + (2\zeta_T \omega_T + 2\zeta_s \omega_s \bar{M} + 2\zeta_R \omega_R)s^3 + (\omega_T^2 + \omega_s^2 \bar{M} + \omega_R^2 + (2\zeta_T \omega_T + 2\zeta_R \omega_R)2\zeta_s \omega_s)s^2 + (2\zeta_s \omega_s (\omega_T^2 + \omega_R^2) + (2\zeta_T \omega_T + 2\zeta_R \omega_R)\omega_s^2)s + (\omega_T^2 + \omega_R^2)\omega_s^2}$$

Find the specimen velocity transfer function relative to table velocity, $\frac{sX_S}{sX_T}(s)$:

$$\begin{aligned} \frac{sX_T}{sX_A}(s) &= \frac{a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}} \\ \frac{sX_S}{sX_A}(s) &= \frac{-a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} \\ \Rightarrow \frac{sX_S}{sX_T}(s) &= \frac{-a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} \cdot \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{22}b_1} = \frac{-a_{21}}{a_{22}} = \frac{2\zeta_s \omega_s s + \omega_s^2}{s^2 + 2\zeta_s \omega_s s + \omega_s^2} \end{aligned}$$

Find the specimen force transfer function relative to table velocity, $\frac{F_s}{sX_T}(s)$:

$$\frac{F_s}{sX_T}(s) = M_s \frac{s^2 X_S}{sX_T} = sM_s \frac{sX_S}{sX_T}$$

Motion and force transfer functions (single translational DOF)

$$\begin{aligned}\frac{sX_T}{sX_A}(s) &= \frac{B_T^V(s)}{A(s)} & \frac{F_T}{sX_A}(s) &= \frac{1}{s} \frac{B_T^F(s)}{A(s)} \\ \frac{sX_S}{sX_A}(s) &= \frac{B_S^V(s)}{A(s)} & \frac{F_S}{sX_A}(s) &= \frac{1}{s} \frac{B_S^F(s)}{A(s)}\end{aligned}$$

where

$$A(s) = s^4 + (2\zeta_T\omega_T + 2\zeta_S\omega_S\bar{M} + 2\zeta_R\omega_T)s^3 + (\omega_T^2 + \omega_S^2\bar{M} + \omega_R^2 + (2\zeta_T\omega_T + 2\zeta_R\omega_T)2\zeta_S\omega_S)s^2 + (2\zeta_S\omega_S(\omega_T^2 + \omega_R^2) + (2\zeta_T\omega_T + 2\zeta_R\omega_T)\omega_S^2)s + (\omega_T^2 + \omega_R^2)\omega_S^2$$

$$\begin{aligned}B_T^V(s) &= (s^2 + 2\zeta_S\omega_Ss + \omega_S^2)(2\zeta_T\omega_Ts + \omega_T^2) \\ &= 2\zeta_T\omega_Ts^3 + (\omega_T^2 + 2\zeta_T\omega_T2\zeta_S\omega_S)s^2 + (2\zeta_T\omega_T\omega_S^2 + 2\zeta_S\omega_S\omega_T^2)s + \omega_T^2\omega_S^2\end{aligned}$$

$$\begin{aligned}B_S^V(s) &= (2\zeta_S\omega_Ss + \omega_S^2) \cdot (2\zeta_T\omega_Ts + \omega_T^2) \\ &= 2\zeta_T\omega_T2\zeta_S\omega_Ss^2 + (2\zeta_T\omega_T\omega_S^2 + 2\zeta_S\omega_S\omega_T^2)s + \omega_T^2\omega_S^2\end{aligned}$$

$$\begin{aligned}B_T^F(s) &= M_T(2\zeta_T\omega_Ts + \omega_T^2) \\ &\quad \cdot s^4 + (2\zeta_S\omega_S\bar{M} + 2\zeta_R\omega_T)s^3 + (\omega_S^2\bar{M} + \omega_R^2 + 2\zeta_S\omega_S2\zeta_R\omega_T)s^2 + (2\zeta_S\omega_S\omega_R^2 + 2\zeta_R\omega_T\omega_S^2)s + \omega_R^2\omega_S^2 \\ &= M_T \cdot \left[\begin{aligned} &2\zeta_T\omega_Ts^5 \\ &+ (\omega_T^2 + 2\zeta_T\omega_T(2\zeta_S\omega_S\bar{M} + 2\zeta_R\omega_T))s^4 \\ &+ (2\zeta_T\omega_T(\omega_S^2\bar{M} + \omega_R^2 + 2\zeta_S\omega_S2\zeta_R\omega_T) + \omega_T^2(2\zeta_S\omega_S\bar{M} + 2\zeta_R\omega_T))s^3 \\ &+ (2\zeta_T\omega_T(2\zeta_S\omega_S\omega_R^2 + 2\zeta_R\omega_T\omega_S^2) + \omega_T^2(\omega_S^2\bar{M} + \omega_R^2 + 2\zeta_S\omega_S2\zeta_R\omega_T))s^2 \\ &+ (2\zeta_T\omega_T\omega_R^2\omega_S^2 + \omega_T^2(2\zeta_S\omega_S\omega_R^2 + 2\zeta_R\omega_T\omega_S^2))s \\ &+ \omega_T^2\omega_S^2\omega_R^2 \end{aligned} \right]\end{aligned}$$

$$\begin{aligned}B_S^F(s) &= s^2M_S(2\zeta_S\omega_Ss + \omega_S^2) \cdot (2\zeta_T\omega_Ts + \omega_T^2) \\ &= s^2M_S(2\zeta_T\omega_T2\zeta_S\omega_Ss^2 + (2\zeta_T\omega_T\omega_S^2 + 2\zeta_S\omega_S\omega_T^2)s + \omega_T^2\omega_S^2)\end{aligned}$$

The mass ratio is defined as

$$\bar{M} = 1 + \frac{M_S}{M_T}$$

Relative specimen motion and force transfer functions (single translational DOF)

$$\begin{aligned}\frac{sX_S}{sX_T}(s) &= \frac{2\zeta_S\omega_Ss + \omega_S^2}{s^2 + 2\zeta_S\omega_Ss + \omega_S^2} \\ \frac{F_S}{sX_T}(s) &= sM_S \frac{sX_S}{sX_T}\end{aligned}$$

Solutions to matrix equations

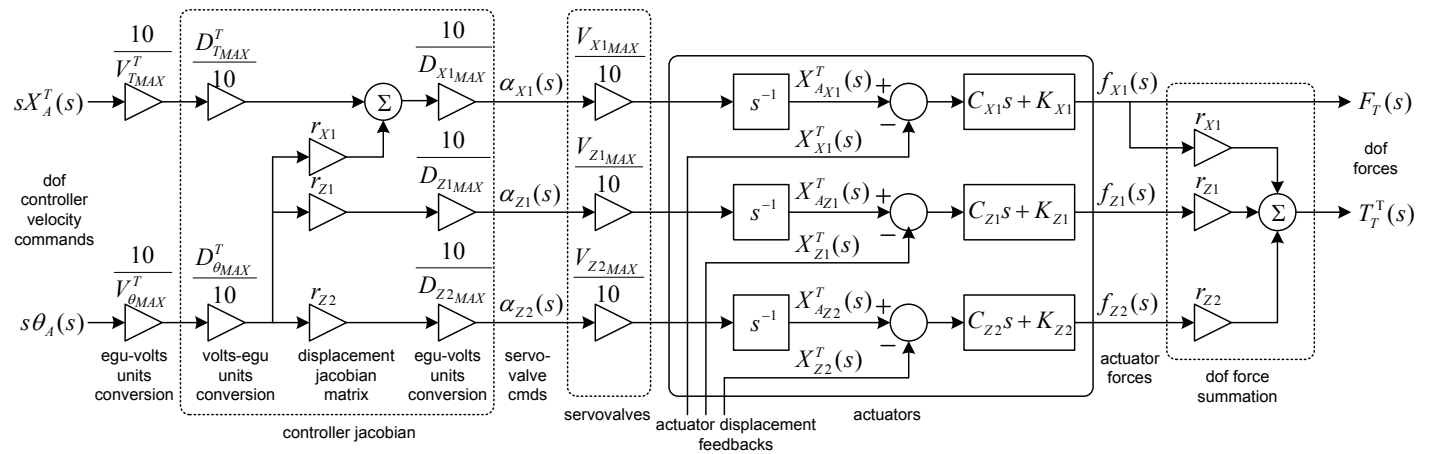
Solution to third-order matrix equation:

$$\begin{aligned}\mathbf{Ax} = \mathbf{By} &\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{By} &= \frac{\begin{bmatrix} a_{22}a_{33} - a_{32}a_{23} & a_{32}a_{13} - a_{12}a_{33} & a_{12}a_{23} - a_{22}a_{13} \\ a_{31}a_{23} - a_{21}a_{33} & a_{11}a_{33} - a_{31}a_{13} & a_{21}a_{13} - a_{11}a_{23} \\ a_{21}a_{32} - a_{31}a_{22} & a_{31}a_{12} - a_{11}a_{32} & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}{a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})} \mathbf{y}\end{aligned}$$

Solution to second-order matrix equation:

$$\begin{aligned}\mathbf{Ax} = \mathbf{By} &\Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ \Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{By} &= \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}{a_{11}a_{22} - a_{12}a_{21}} \mathbf{y}\end{aligned}$$

Physical System Force Equations



Variables

$X_{AX1}, X_{AZ1}, X_{AZ2}$	Actuator oil spring displacement
X_{X1}, X_{Z1}, X_{Z2}	Actuator displacement
C_{X1}, C_{Z1}, C_{Z2}	Actuator damping
K_{X1}, K_{Z1}, K_{Z2}	Actuator spring constant
r_{X1}	Vertical distance from table origin to horizontal actuator X1 (same as r_{TA})
r_{Z1}, r_{Z2}	Horizontal distance from table origin to vertical actuators Z1 & Z2
D_{TMAX}^T	Fullscale table translation and translational velocity units
$D_{\theta MAX}^T, V_{\theta MAX}^T$	Fullscale table rotation and rotational velocity units
D_{X1MAX}, V_{X1MAX}	Fullscale horizontal actuator translation and translational velocity units
D_{Z1MAX}, V_{Z1MAX}	Fullscale vertical actuator translation and translational velocity units
D_{Z2MAX}, V_{Z2MAX}	Fullscale vertical actuator translation and translational velocity units

Actuator servovalve commands:

$$\alpha_{x_1}(s) = \frac{D_{T_{MAX}}^T}{D_{x_1_{MAX}}^T} \frac{10}{V_{T_{MAX}}^T} s X_A^T(s) + r_{x_1} \frac{D_{\theta_{MAX}}^T}{D_{x_1_{MAX}}^T} \frac{10}{V_{\theta_{MAX}}^T} s \theta_A(s)$$

$$\alpha_{z1}(s) = r_{z1} \frac{D_{\theta_{MAX}}^T}{D_{z1_{MAX}}} \frac{10}{V_{\theta_{MAX}}^T} s \theta_A(s)$$

$$\alpha_{z_2}(s) = r_{z_2} \frac{D_{\theta_{MAX}}^T}{D_{z_2_{MAX}}} \frac{10}{V_{\theta_{MAX}}^T} s \theta_A(s)$$

Actuator oil spring displacements:

$$X_{A_{X_1}}^T(s) = \frac{V_{X_1MAX}}{10} \frac{1}{s} \alpha_{X_1}(s) = \frac{V_{X_1MAX}}{10} \frac{1}{s} \left(\frac{D_{TMAX}^T}{D_{X_1MAX}} \frac{10}{V_{TMAX}^T} s X_A^T(s) + r_{X_1} \frac{D_{\theta MAX}^T}{D_{X_1MAX}} \frac{10}{V_{\theta MAX}^T} s \theta_A(s) \right)$$

$$\Rightarrow X_{A_{X1}}^T(s) = \frac{D_{T_{MAX}}^T}{D_{X1_{MAX}}} \frac{V_{X1_{MAX}}}{V_{T_{MAX}}^T} X_A^T(s) + r_{X1} \frac{D_{\theta_{MAX}}^T}{D_{X1_{MAX}}} \frac{V_{X1_{MAX}}}{V_{\theta_{MAX}}^T} \theta_A(s)$$

$$X_{A_{Z1}}^T(s) = \frac{V_{Z1_{MAX}}}{10} \frac{1}{s} \alpha_{Z1}(s) = \frac{V_{Z1_{MAX}}}{10} \frac{1}{s} \left(r_{Z1} \frac{D_{\theta_{MAX}}^T}{D_{Z1_{MAX}}} \frac{10}{V_{\theta_{MAX}}^T} s \theta_A(s) \right)$$

$$\Rightarrow X_{A_{Z1}}^T(s) = r_{Z1} \frac{D_{\theta_{MAX}}^T}{D_{Z1_{MAX}}} \frac{V_{Z1_{MAX}}}{V_{\theta_{MAX}}^T} \theta_A(s)$$

$$X_{A_{Z2}}^T(s) = \frac{V_{Z2_{MAX}}}{10} \frac{1}{s} \alpha_{Z2}(s) = \frac{V_{Z2_{MAX}}}{10} \frac{1}{s} \left(r_{Z2} \frac{D_{\theta_{MAX}}^T}{D_{Z2_{MAX}}} \frac{10}{V_{\theta_{MAX}}^T} s \theta_A(s) \right)$$

$$\Rightarrow X_{A_{Z2}}^T(s) = r_{Z2} \frac{D_{\theta_{MAX}}^T}{D_{Z2_{MAX}}} \frac{V_{Z2_{MAX}}}{V_{\theta_{MAX}}^T} \theta_A(s)$$

Actuator feedbacks:

$$X_{X1}^T(s) = X_T^T(s) + r_{X1} \theta_T(s)$$

$$X_{Z1}^T(s) = r_{Z1} \theta_T(s)$$

$$X_{Z2}^T(s) = r_{Z2} \theta_T(s)$$

Actuator forces:

$$f_{X1}(s) = [C_{X1}s + K_{X1}] \cdot [X_{A_{X1}}^T(s) - X_{X1}^T(s)]$$

$$\Rightarrow f_{X1}(s) = [C_{X1}s + K_{X1}] \cdot \left[\frac{D_{T_{MAX}}^T}{D_{X1_{MAX}}} \frac{V_{X1_{MAX}}}{V_{T_{MAX}}^T} X_A^T(s) + r_{X1} \frac{D_{\theta_{MAX}}^T}{D_{X1_{MAX}}} \frac{V_{X1_{MAX}}}{V_{\theta_{MAX}}^T} \theta_A(s) - (X_T^T(s) + r_{X1} \theta_T(s)) \right]$$

$$\Rightarrow f_{X1}(s) = [C_{X1}s + K_{X1}] \cdot \left[\frac{D_{T_{MAX}}^T}{D_{X1_{MAX}}} \frac{V_{X1_{MAX}}}{V_{T_{MAX}}^T} X_A^T(s) - X_T^T(s) + r_{X1} \left(\frac{D_{\theta_{MAX}}^T}{D_{X1_{MAX}}} \frac{V_{X1_{MAX}}}{V_{\theta_{MAX}}^T} \theta_A(s) - \theta_T(s) \right) \right]$$

$$f_{Z1}(s) = [C_{Z1}s + K_{Z1}] \cdot [X_{A_{Z1}}^T(s) - X_{Z1}^T(s)]$$

$$\Rightarrow f_{Z1}(s) = [C_{Z1}s + K_{Z1}] \cdot \left[r_{Z1} \frac{D_{\theta_{MAX}}^T}{D_{Z1_{MAX}}} \frac{V_{Z1_{MAX}}}{V_{\theta_{MAX}}^T} \theta_A(s) - r_{Z1} \theta_T(s) \right]$$

$$\Rightarrow f_{Z1}(s) = r_{Z1} [C_{Z1}s + K_{Z1}] \cdot \left[\frac{D_{\theta_{MAX}}^T}{D_{Z1_{MAX}}} \frac{V_{Z1_{MAX}}}{V_{\theta_{MAX}}^T} \theta_A(s) - \theta_T(s) \right]$$

$$f_{Z2}(s) = [C_{Z2}s + K_{Z2}] \cdot [X_{A_{Z2}}^T(s) - X_{Z2}^T(s)]$$

$$\Rightarrow f_{Z2}(s) = [C_{Z2}s + K_{Z2}] \cdot \left[r_{Z2} \frac{D_{\theta_{MAX}}^T}{D_{Z2_{MAX}}} \frac{V_{Z2_{MAX}}}{V_{\theta_{MAX}}^T} \theta_A(s) - r_{Z2} \theta_T(s) \right]$$

$$\Rightarrow f_{Z2}(s) = r_{Z2} [C_{Z2}s + K_{Z2}] \cdot \left[\frac{D_{\theta_{MAX}}^T}{D_{Z2_{MAX}}} \frac{V_{Z2_{MAX}}}{V_{\theta_{MAX}}^T} \theta_A(s) - \theta_T(s) \right]$$

The actuator forces above can be simplified by noting that it is usually the case that

$$\frac{D_{T_{MAX}}^T}{D_{X1_{MAX}}} \frac{V_{X1_{MAX}}}{V_{T_{MAX}}^T} \approx \frac{D_{\theta_{MAX}}^T}{D_{X1_{MAX}}} \frac{V_{X1_{MAX}}}{V_{\theta_{MAX}}^T} \approx 1$$

$$\frac{D_{\theta_{MAX}}^T}{D_{Z1_{MAX}}} \frac{V_{Z1_{MAX}}}{V_{\theta_{MAX}}^T} \approx \frac{D_{\theta_{MAX}}^T}{D_{Z2_{MAX}}} \frac{V_{Z2_{MAX}}}{V_{\theta_{MAX}}^T} \approx 1$$

yielding

$$f_{X1}(s) = [C_{X1}s + K_{X1}] \cdot [X_A^T(s) - X_T^T(s) + r_{X1}(\theta_A(s) - \theta_T(s))]$$

$$f_{Z1}(s) = r_{Z1} [C_{Z1}s + K_{Z1}] \cdot [\theta_A(s) - \theta_T(s)]$$

$$f_{Z2}(s) = r_{Z2} [C_{Z2}s + K_{Z2}] \cdot [\theta_A(s) - \theta_T(s)]$$

Linear DOF force $F_T(s)$:

$$F_T(s) = f_{X1}$$

$$\Rightarrow F_T(s) = [C_{X1}s + K_{X1}] \cdot [X_A^T(s) - X_T^T(s) + r_{X1}(\theta_A(s) - \theta_T(s))]$$

Noting that the vertical distance from table origin to horizontal prime mover is computed as

$$\boxed{-r_{AT} = r_{TA} = r_{X1}}$$

and the linear DOF damping and spring constants at the CG as a function of actuator damping and spring constants are computed as

$$\boxed{C_T^G = C_{X1}} \quad \text{and} \quad \boxed{K_T^G = K_{X1}},$$

the linear DOF force is therefore

$$\boxed{F_T(s) = [C_T^G s + K_T^G] \cdot [X_A^T(s) - X_T^T(s) - r_{AT}(\theta_A(s) - \theta_T(s))]}$$

which is exactly the $F_T(s)$ force equation in the *Equations of motion at the table origin* section.

Rotational DOF force $T_T^T(s)$:

$$T_T^T(s) = r_{X1}f_{X1}(s) + r_{Z1}f_{Z1}(s) + r_{Z2}f_{Z2}(s)$$

$$\begin{aligned}
T_T^T(s) &= r_{X1} [C_{X1}s + K_{X1}] \cdot [X_A^T(s) - X_T^T(s) + r_{X1}(\theta_A(s) - \theta_T(s))] \\
\Rightarrow &+ r_{Z1}^2 [C_{Z1}s + K_{Z1}] \cdot [\theta_A(s) - \theta_T(s)] \\
&+ r_{Z2}^2 [C_{Z2}s + K_{Z2}] \cdot [\theta_A(s) - \theta_T(s)] \\
\\
T_T^T(s) &= r_{X1} [C_{X1}s + K_{X1}] \cdot [X_A^T(s) - X_T^T(s)] \\
\Rightarrow &+ r_{X1}^2 [C_{X1}s + K_{X1}] \cdot [\theta_A(s) - \theta_T(s)] \\
&+ r_{Z1}^2 [C_{Z1}s + K_{Z1}] \cdot [\theta_A(s) - \theta_T(s)] \\
&+ r_{Z2}^2 [C_{Z2}s + K_{Z2}] \cdot [\theta_A(s) - \theta_T(s)] \\
\\
\Rightarrow T_T^T(s) &= \left[(r_{X1}^2 C_{X1} + r_{Z1}^2 C_{Z1} + r_{Z2}^2 C_{Z2})s + (r_{X1}^2 K_{X1} + r_{Z1}^2 K_{Z1} + r_{Z2}^2 K_{Z2}) \right] \cdot [\theta_A(s) - \theta_T(s)] \\
&+ r_{X1} [C_{X1}s + K_{X1}] \cdot [X_A^T(s) - X_T^T(s)]
\end{aligned}$$

Noting that the vertical distance from table origin to horizontal prime mover is computed as

$$\boxed{-r_{AT} = r_{TA} = r_{X1}}$$

and the linear and rotational DOF damping and spring constants at the CG as a function of actuator damping and spring constants are computed as

$$\boxed{C_T^G = C_{X1}} \quad \text{and} \quad \boxed{K_T^G = K_{X1}}$$

$$\boxed{C_{\theta_T}^G = r_{AG}^2 C_{X1} + r_{Z1}^2 C_{Z1} + r_{Z2}^2 C_{Z2}} \quad \text{and} \quad \boxed{K_{\theta_T}^G = r_{AG}^2 K_{X1} + r_{Z1}^2 K_{Z1} + r_{Z2}^2 K_{Z2}},$$

find expressions for $r_{X1}^2 C_{X1} + r_{Z1}^2 C_{Z1} + r_{Z2}^2 C_{Z2}$ and $r_{X1}^2 K_{X1} + r_{Z1}^2 K_{Z1} + r_{Z2}^2 K_{Z2}$:

$$\begin{aligned}
&r_{X1}^2 C_{X1} + r_{Z1}^2 C_{Z1} + r_{Z2}^2 C_{Z2} \\
&= r_{AT}^2 C_{X1} + r_{Z1}^2 C_{Z1} + r_{Z2}^2 C_{Z2} \\
&= r_{AG}^2 C_{X1} + r_{Z1}^2 C_{Z1} + r_{Z2}^2 C_{Z2} + r_{AT}^2 C_{X1} - r_{AG}^2 C_{X1} \\
&= (r_{AG}^2 C_{X1} + r_{Z1}^2 C_{Z1} + r_{Z2}^2 C_{Z2}) - (r_{AG}^2 - r_{AT}^2) C_{X1} \\
&= C_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) C_T^G \\
\Rightarrow &\boxed{r_{X1}^2 C_{X1} + r_{Z1}^2 C_{Z1} + r_{Z2}^2 C_{Z2} = C_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) C_T^G}
\end{aligned}$$

Likewise,

$$\boxed{r_{X1}^2 K_{X1} + r_{Z1}^2 K_{Z1} + r_{Z2}^2 K_{Z2} = K_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) K_T^G}$$

The rotational DOF force is therefore

$$\boxed{T_T^T(s) = \left[(C_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) C_T^G)s + (K_{\theta_T}^G - (r_{AG}^2 - r_{AT}^2) K_T^G) \right] \cdot [\theta_A(s) - \theta_T(s)] - r_{AT} [C_T^G s + K_T^G] \cdot [X_A^T(s) - X_T^T(s)]}$$

which is exactly the $T_T^T(s)$ force equation in the *Equations of motion at the table origin* section.