

State-Space Formulation of Multi-Degree-of-Freedom Dynamics

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In this document, the dynamics of a multi-degree-of-freedom (MDOF) system will be derived in state-space form from the system's mass, damping, and stiffness matrices. State-space formulation has the following benefits:

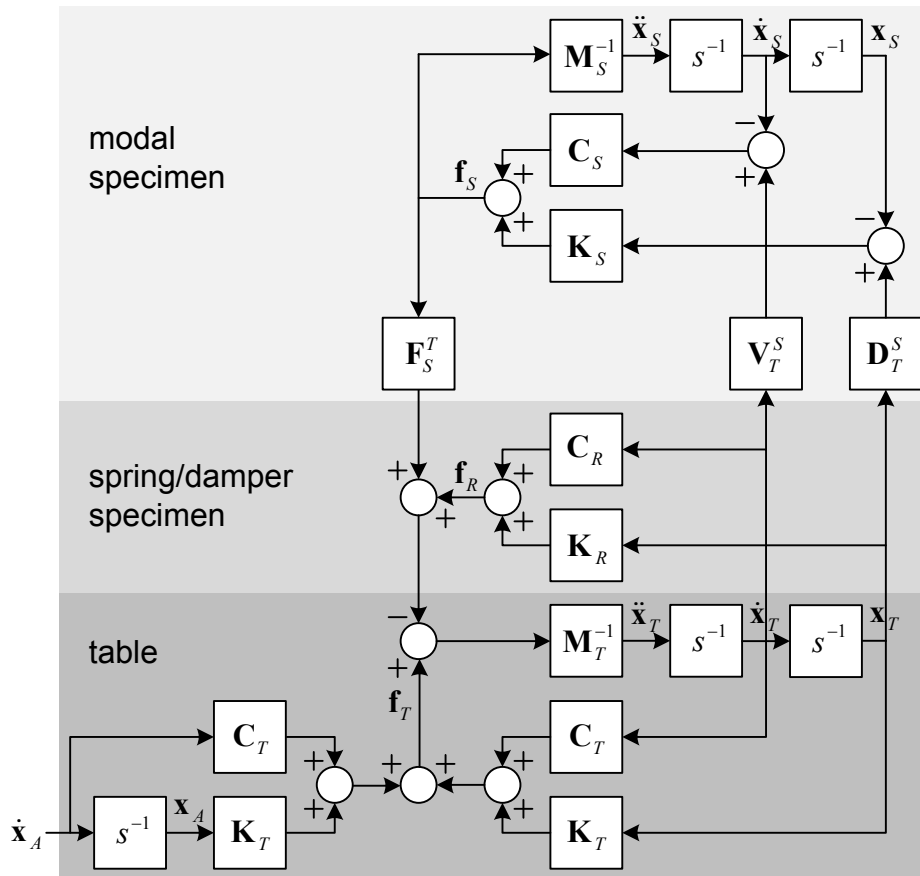
- 1) Frequency response functions (FRFs) are easily computed from the state-space matrices.
- 2) Eigenanalysis of the state-space feedback matrix yields modal frequencies and shapes.

Both open-loop and closed-loop state-space equations will be derived. Sections are included that show how to determine the mass, damping, and stiffness matrices.

To simplify the analysis, all quantities are assumed to be in canonical units, i.e., units containing only linear displacement, linear force, radians, and time.

Derivation of State-Space Equations: Open-loop Dynamics

A block diagram of the MDOF system (in this case, an MDOF shake table) is shown below:



The system is composed of three subsystems: the table, a spring/damper specimen, and a modal specimen. The input to the table subsystem is vector $\dot{\mathbf{x}}_A$, which represents the oilspring velocity of idealized hydraulic “DOF actuators”, and the outputs are vectors representing table displacement, velocity, acceleration, and force $\mathbf{x}_T$, $\dot{\mathbf{x}}_T$, $\ddot{\mathbf{x}}_T$, and $\mathbf{f}_T$. Mass/damper/spring matrices $\mathbf{M}_T$, $\mathbf{C}_T$, and $\mathbf{K}_T$ define the physical characteristics of the table subsystem. Note that the subscript “T” is used not only to identify the table subsystem, but also to denote that the item is defined with respect to origin of the table coordinate frame. The inputs to the modal specimen subsystem are table displacement/velocity vectors $\mathbf{x}_T$ and $\dot{\mathbf{x}}_T$ and the outputs are modal specimen displacement, velocity, acceleration, and force vectors $\mathbf{x}_S$, $\dot{\mathbf{x}}_S$, $\ddot{\mathbf{x}}_S$, and $\mathbf{f}_S$, while mass/damper/spring matrices $\mathbf{M}_S$, $\mathbf{C}_S$, and $\mathbf{K}_S$ define the physical characteristics. Force, velocity, and displacement transforms $\mathbf{F}_S^T$, $\mathbf{V}_T^S$, and $\mathbf{D}_T^S$ transform force and motion between the table coordinate frame and the modal specimen frame. The inputs to the spring/damper subsystem are table displacement/velocity vectors $\mathbf{x}_T$ and $\dot{\mathbf{x}}_T$ and the output is force vector $\mathbf{f}_R$, while damper/spring matrices $\mathbf{C}_T$ and $\mathbf{K}_T$ define the physical characteristics.

Derivation of State-Space Equations: Open-loop Dynamics

The differential equations that describe the dynamics of this system are

$$\begin{aligned}\mathbf{M}_T \ddot{\mathbf{x}}_T &= \mathbf{f}_T - \mathbf{F}_S^T \mathbf{f}_S - \mathbf{f}_R \\ \mathbf{M}_S \ddot{\mathbf{x}}_S &= \mathbf{f}_S\end{aligned}$$

where

$$\begin{aligned}\mathbf{f}_T &= \mathbf{C}_T (\dot{\mathbf{x}}_A - \dot{\mathbf{x}}_T) + \mathbf{K}_T (\mathbf{x}_A - \mathbf{x}_T) \\ \mathbf{f}_S &= \mathbf{C}_S (\mathbf{V}_T^S \dot{\mathbf{x}}_T - \dot{\mathbf{x}}_S) + \mathbf{K}_S (\mathbf{D}_T^S \mathbf{x}_T - \mathbf{x}_S) \\ \mathbf{f}_R &= \mathbf{C}_R \dot{\mathbf{x}}_T + \mathbf{K}_R \mathbf{x}_T\end{aligned}$$

Combining these equations,

$$\begin{aligned}\mathbf{M}_T \ddot{\mathbf{x}}_T &= \mathbf{C}_T (\dot{\mathbf{x}}_A - \dot{\mathbf{x}}_T) + \mathbf{K}_T (\mathbf{x}_A - \mathbf{x}_T) - \mathbf{F}_S^T \mathbf{C}_S (\mathbf{V}_T^S \dot{\mathbf{x}}_T - \dot{\mathbf{x}}_S) - \mathbf{F}_S^T \mathbf{K}_S (\mathbf{D}_T^S \mathbf{x}_T - \mathbf{x}_S) - \mathbf{C}_R \dot{\mathbf{x}}_T - \mathbf{K}_R \mathbf{x}_T \\ \mathbf{M}_S \ddot{\mathbf{x}}_S &= \mathbf{C}_S (\mathbf{V}_T^S \dot{\mathbf{x}}_T - \dot{\mathbf{x}}_S) + \mathbf{K}_S (\mathbf{D}_T^S \mathbf{x}_T - \mathbf{x}_S) \\ \Rightarrow \mathbf{M}_T \ddot{\mathbf{x}}_T &= \mathbf{K}_T \mathbf{x}_A - (\mathbf{K}_T + \mathbf{F}_S^T \mathbf{K}_S \mathbf{D}_T^S + \mathbf{K}_R) \mathbf{x}_T - (\mathbf{C}_T + \mathbf{F}_S^T \mathbf{C}_S \mathbf{V}_T^S + \mathbf{C}_R) \dot{\mathbf{x}}_T + \mathbf{F}_S^T \mathbf{K}_S \mathbf{x}_S + \mathbf{F}_S^T \mathbf{C}_S \dot{\mathbf{x}}_S + \mathbf{C}_T \dot{\mathbf{x}}_A \\ \Rightarrow \mathbf{M}_S \ddot{\mathbf{x}}_S &= \mathbf{0} \cdot \mathbf{x}_A + \mathbf{K}_S \mathbf{D}_T^S \mathbf{x}_T + \mathbf{C}_S \mathbf{V}_T^S \dot{\mathbf{x}}_T - \mathbf{K}_S \mathbf{x}_S - \mathbf{C}_S \dot{\mathbf{x}}_S + \mathbf{0} \cdot \dot{\mathbf{x}}_A \\ \Rightarrow \begin{bmatrix} \mathbf{M}_T \ddot{\mathbf{x}}_T \\ \mathbf{M}_S \ddot{\mathbf{x}}_S \end{bmatrix} &= \begin{bmatrix} \mathbf{K}_T & -(\mathbf{K}_T + \mathbf{K}_S^T + \mathbf{K}_R) & -(\mathbf{C}_T + \mathbf{C}_S^T + \mathbf{C}_R) & \mathbf{F}_S^T \mathbf{K}_S & \mathbf{F}_S^T \mathbf{C}_S \\ \mathbf{0} & \mathbf{K}_S \mathbf{D}_T^S & \mathbf{C}_S \mathbf{V}_T^S & -\mathbf{K}_S & -\mathbf{C}_S \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_T \\ \dot{\mathbf{x}}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_S \end{bmatrix} + \begin{bmatrix} \mathbf{C}_T \\ \mathbf{0} \end{bmatrix} \dot{\mathbf{x}}_A\end{aligned}$$

where the modal specimen damping and stiffness has been transformed to the table origin using

$$\begin{aligned}\mathbf{C}_S^T &= \mathbf{F}_S^T \mathbf{C}_S \mathbf{V}_T^S \\ \mathbf{K}_S^T &= \mathbf{F}_S^T \mathbf{K}_S \mathbf{D}_T^S\end{aligned}$$

These differential equations can now be converted to state-space form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

where $\mathbf{A}$ is the state matrix, $\mathbf{B}$ is the input matrix, $\mathbf{C}$ is the output matrix, and $\mathbf{D}$ is the feedthrough matrix.

Defining the input vector $\mathbf{u}$ as oilspring velocity of idealized DOF actuators

$$\mathbf{u} = \dot{\mathbf{x}}_A$$

and the state vector $\mathbf{x}$ as comprising oilspring displacement $\mathbf{x}_A$, table displacement $\mathbf{x}_T$, modal specimen displacement $\mathbf{x}_S$, table velocity $\dot{\mathbf{x}}_T$, and modal specimen velocity $\dot{\mathbf{x}}_S$

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_T \\ \dot{\mathbf{x}}_S \end{bmatrix},$$

the differential equations can be rearranged into state-space form as

$$\begin{bmatrix} \dot{\mathbf{x}}_A \\ \dot{\mathbf{x}}_T \\ \ddot{\mathbf{x}}_T \\ \dot{\mathbf{x}}_S \\ \ddot{\mathbf{x}}_S \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_T^{-1}\mathbf{K}_T & -\mathbf{M}_T^{-1}(\mathbf{K}_T + \mathbf{K}_S^T + \mathbf{K}_R) & -\mathbf{M}_T^{-1}(\mathbf{C}_T + \mathbf{C}_S^T + \mathbf{C}_R) & \mathbf{M}_T^{-1}\mathbf{F}_S^T\mathbf{K}_S & \mathbf{M}_T^{-1}\mathbf{F}_S^T\mathbf{C}_S \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{M}_S^{-1}\mathbf{K}_S\mathbf{D}_T^S & \mathbf{M}_S^{-1}\mathbf{C}_S\mathbf{V}_T^S & -\mathbf{M}_S^{-1}\mathbf{K}_S & -\mathbf{M}_S^{-1}\mathbf{C}_S \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_T \\ \dot{\mathbf{x}}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_S \end{bmatrix} + \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \dot{\mathbf{x}}_A$$

$$\Rightarrow \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

Table force $\mathbf{f}_T$ and modal specimen force $\mathbf{f}_S$ are given by

$$\begin{aligned} \mathbf{f}_T &= \mathbf{K}_T\mathbf{x}_A + \mathbf{C}_T\dot{\mathbf{x}}_A - \mathbf{K}_T\mathbf{x}_T - \mathbf{C}_T\dot{\mathbf{x}}_T \\ \mathbf{f}_S &= \mathbf{K}_S\mathbf{D}_T^S\mathbf{x}_T + \mathbf{C}_S\mathbf{V}_T^S\dot{\mathbf{x}}_T - \mathbf{K}_S\mathbf{x}_S - \mathbf{C}_S\dot{\mathbf{x}}_S \end{aligned}$$

Defining the output vector $\mathbf{y}$ as comprising system motion and force states

$$\mathbf{y} = \begin{bmatrix} \mathbf{x}_T \\ \dot{\mathbf{x}}_T \\ \mathbf{f}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_S \\ \mathbf{f}_S \end{bmatrix},$$

the system output equation can be written as

$$\mathbf{y} = \begin{bmatrix} \mathbf{x}_T \\ \dot{\mathbf{x}}_T \\ \mathbf{f}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_S \\ \mathbf{f}_S \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_T & -\mathbf{K}_T & -\mathbf{C}_T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{K}_S\mathbf{D}_T^S & \mathbf{C}_S\mathbf{V}_T^S & -\mathbf{K}_S & -\mathbf{C}_S \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_T \\ \dot{\mathbf{x}}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_S \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{C}_T \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \dot{\mathbf{x}}_A$$

$$\Rightarrow \mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

Summary: Open-Loop Dynamics

Given mass, damping, and stiffness matrices

$\mathbf{M}_T, \mathbf{C}_T, \mathbf{K}_T$	table mass, damping, stiffness (table coordinate frame)
$\mathbf{M}_S, \mathbf{C}_S, \mathbf{K}_S$	modal specimen mass, damping, stiffness (modal coordinate frame)
$\mathbf{F}_S^T, \mathbf{V}_T^S, \mathbf{D}_T^S$	transforms between table and modal coordinate frames
$\mathbf{C}_S^T = \mathbf{F}_S^T \mathbf{C}_S \mathbf{V}_T^S$	modal specimen damping (table coordinate frame)
$\mathbf{K}_S^T = \mathbf{F}_S^T \mathbf{K}_S \mathbf{D}_T^S$	modal specimen stiffness (table coordinate frame)
$\mathbf{C}_R, \mathbf{K}_R$	spring/damper specimen damping, stiffness (table coordinate frame)

the state-space formulation of the open-loop MDOF system dynamics is

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

state transition equation
output equation

where

$$\begin{aligned}\mathbf{u} &= \dot{\mathbf{x}}_A && \text{input: } \dot{\mathbf{x}}_A: \text{ oil velocity} \\ \mathbf{x} &= \begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_T \\ \dot{\mathbf{x}}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_S \end{bmatrix} && \begin{aligned} \text{state: } & \mathbf{x}_A: \text{ oil displacement} \\ & \mathbf{x}_T: \text{ table displacement} \\ & \dot{\mathbf{x}}_T: \text{ table velocity} \\ & \mathbf{x}_S: \text{ modal specimen displacement} \\ & \dot{\mathbf{x}}_S: \text{ modal specimen velocity} \end{aligned} \\ \mathbf{y} &= \begin{bmatrix} \mathbf{x}_T \\ \dot{\mathbf{x}}_T \\ \mathbf{f}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_S \\ \mathbf{f}_S \end{bmatrix} && \begin{aligned} \text{output: } & \mathbf{x}_T: \text{ table displacement} \\ & \dot{\mathbf{x}}_T: \text{ table velocity} \\ & \mathbf{f}_T: \text{ table force} \\ & \mathbf{x}_S: \text{ modal specimen displacement} \\ & \dot{\mathbf{x}}_S: \text{ modal specimen velocity} \\ & \mathbf{f}_S: \text{ modal specimen force} \end{aligned}\end{aligned}$$

Note: Modal specimen motion states $\mathbf{x}_S$ and $\dot{\mathbf{x}}_S$ contain only translational DOFs; rotational DOF motion states are zero. This is not the case with modal specimen force output $\mathbf{f}_S$, which has both translational and rotational DOFs.

(State-space matrices appear on the following page.)

Summary: Open-Loop Dynamics (continued)

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_T^{-1}\mathbf{K}_T & -\mathbf{M}_T^{-1}(\mathbf{K}_T + \mathbf{K}_S^T + \mathbf{K}_R) & -\mathbf{M}_T^{-1}(\mathbf{C}_T + \mathbf{C}_S^T + \mathbf{C}_R) & \mathbf{M}_T^{-1}\mathbf{F}_S^T\mathbf{K}_S & \mathbf{M}_T^{-1}\mathbf{F}_S^T\mathbf{C}_S \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{M}_S^{-1}\mathbf{K}_S\mathbf{D}_T^S & \mathbf{M}_S^{-1}\mathbf{C}_S\mathbf{V}_T^S & -\mathbf{M}_S^{-1}\mathbf{K}_S & -\mathbf{M}_S^{-1}\mathbf{C}_S \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{M}_T^{-1}\mathbf{C}_T \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}_T & -\mathbf{K}_T & -\mathbf{C}_T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{K}_S\mathbf{D}_T^S & \mathbf{C}_S\mathbf{V}_T^S & -\mathbf{K}_S & -\mathbf{C}_S \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{C}_T \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Note: If there is no modal specimen, delete rows and columns associated with $\mathbf{x}_S$, $\dot{\mathbf{x}}_S$, and $\mathbf{f}_S$.

Derivation of State-Space Equations: Closed-loop Dynamics

The presence of a feedback controller alters the state-space matrices. Find the closed loop state-space matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$, and $\tilde{\mathbf{D}}$ assuming a controller feedback gain matrix $\mathbf{K}_C$, reference input $\mathbf{r}$, system input $\mathbf{u}$, and system output $\mathbf{y}$.

The controller equation is

$$\mathbf{u} = \mathbf{r} - \mathbf{K}_C \mathbf{y}$$

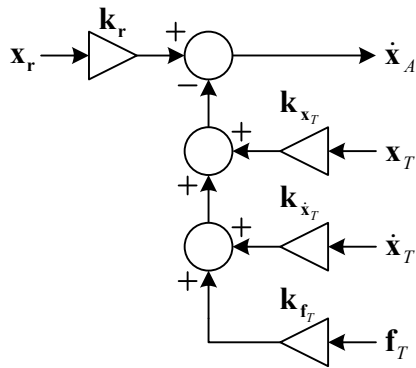
Inserting the expression for $\mathbf{u}$ above into the output equation,

$$\begin{aligned} \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} = \mathbf{C}\mathbf{x} + \mathbf{D}(\mathbf{r} - \mathbf{K}_C \mathbf{y}) \\ \Rightarrow (\mathbf{I} + \mathbf{D}\mathbf{K}_C) \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{r} \\ \Rightarrow \mathbf{y} &= (\mathbf{I} + \mathbf{D}\mathbf{K}_C)^{-1} \mathbf{C}\mathbf{x} + (\mathbf{I} + \mathbf{D}\mathbf{K}_C)^{-1} \mathbf{D}\mathbf{r} \\ \Rightarrow \mathbf{y} &= \tilde{\mathbf{C}}\mathbf{x} + \tilde{\mathbf{D}}\mathbf{r} \end{aligned}$$

Inserting the expression for $\mathbf{y}$ above into state transition equation,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} = \mathbf{A} + \mathbf{B}(\mathbf{r} - \mathbf{K}_C \mathbf{y}) = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}_C \mathbf{y} + \mathbf{B}\mathbf{r} \\ \Rightarrow \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}_C (\tilde{\mathbf{C}}\mathbf{x} + \tilde{\mathbf{D}}\mathbf{r}) + \mathbf{B}\mathbf{r} \\ \Rightarrow \dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{B}\mathbf{K}_C \tilde{\mathbf{C}}) \mathbf{x} + (\mathbf{B} - \mathbf{B}\mathbf{K}_C \tilde{\mathbf{D}}) \mathbf{r} \\ \Rightarrow \dot{\mathbf{x}} &= \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{B}}\mathbf{r} \end{aligned}$$

In general, most of the elements of feedback gain matrix $\mathbf{K}_C$ are zero. Find the internal structure of $\mathbf{K}_C$ that implements the following MDOF controller is topology, which is feedback-only (no feedforward):



Note that in the above diagram, the gain elements perform one-to-one multiplication of individual vector elements, not matrix multiplication. In the following, the operator $\otimes$ signifies element multiplication.

$$\dot{\mathbf{x}}_A = (\mathbf{k}_r \otimes \mathbf{x}_r) - (\mathbf{k}_{x_T} \otimes \mathbf{x}_T) - (\mathbf{k}_{\dot{x}_T} \otimes \dot{\mathbf{x}}_T) - (\mathbf{k}_{f_T} \otimes \mathbf{f}_T)$$

$$\Rightarrow \dot{\mathbf{x}}_A = \text{diag}(\mathbf{k}_r) \cdot \mathbf{x}_r - \text{diag}[\mathbf{k}_{x_T} \quad \mathbf{k}_{\dot{x}_T} \quad \mathbf{k}_{f_T} \quad \mathbf{0} \quad \mathbf{0}] \cdot \begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_T \\ \dot{\mathbf{x}}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_S \end{bmatrix}$$

$$\Rightarrow \mathbf{u} = \mathbf{r} - \mathbf{K}_C \mathbf{y}$$

The value of reference gain vector $\mathbf{k}_r$ depends on the controlled variable:

$\mathbf{k}_r = \mathbf{k}_{x_T}$ if the controlled variable is displacement $\mathbf{x}_T$

$\mathbf{k}_r = \mathbf{k}_{\dot{x}_T}$ if the controlled variable is velocity $\dot{\mathbf{x}}_T$

$\mathbf{k}_r = \mathbf{k}_{f_T}$ if the controlled variable is force $\mathbf{f}_T$

Summary: Closed-Loop Dynamics

Given open-loop state-space matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, and feedback gain matrix $\mathbf{K}$, the state-space formulation of the closed-loop MDOF system dynamics is

$$\dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{B}}\mathbf{r} \quad \text{state transition equation}$$

$$\mathbf{y} = \tilde{\mathbf{C}}\mathbf{x} + \tilde{\mathbf{D}}\mathbf{r} \quad \text{output equation}$$

$$\mathbf{u} = \mathbf{r} - \mathbf{K}_C\mathbf{y} \quad \text{controller equation}$$

where

$$\mathbf{r} = \text{diag}(\mathbf{k}_r) \cdot \mathbf{x}_r \quad \text{reference input}$$

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_A \\ \mathbf{x}_T \\ \dot{\mathbf{x}}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_S \end{bmatrix} \quad \begin{array}{ll} \text{state:} & \mathbf{x}_A: \text{ oil displacement} \\ & \mathbf{x}_T: \text{ table displacement} \\ & \dot{\mathbf{x}}_T: \text{ table velocity} \\ & \mathbf{x}_S: \text{ modal specimen displacement} \\ & \dot{\mathbf{x}}_S: \text{ modal specimen velocity} \end{array}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{x}_T \\ \dot{\mathbf{x}}_T \\ \mathbf{f}_T \\ \mathbf{x}_S \\ \dot{\mathbf{x}}_S \\ \mathbf{f}_S \end{bmatrix} \quad \begin{array}{ll} \text{output:} & \mathbf{x}_T: \text{ table displacement} \\ & \dot{\mathbf{x}}_T: \text{ table velocity} \\ & \mathbf{f}_T: \text{ table force} \\ & \mathbf{x}_S: \text{ modal specimen displacement} \\ & \dot{\mathbf{x}}_S: \text{ modal specimen velocity} \\ & \mathbf{f}_S: \text{ modal specimen force} \end{array}$$

$$\begin{aligned} \tilde{\mathbf{A}} &= \mathbf{A} - \mathbf{BK}_C\tilde{\mathbf{C}} & \tilde{\mathbf{B}} &= \mathbf{B} - \mathbf{BK}_C\tilde{\mathbf{D}} \\ \tilde{\mathbf{C}} &= (\mathbf{I} + \mathbf{DK}_C)^{-1}\mathbf{C} & \tilde{\mathbf{D}} &= (\mathbf{I} + \mathbf{DK}_C)^{-1}\mathbf{D} \end{aligned}$$

The feedback gain matrix $\mathbf{K}_C$ is formed from individual feedback gain vectors $\mathbf{k}_{x_T}$, $\mathbf{k}_{\dot{x}_T}$, $\mathbf{k}_{f_T}$

$$\mathbf{K}_C = \text{diag}[\mathbf{k}_{x_T} \quad \mathbf{k}_{\dot{x}_T} \quad \mathbf{k}_{f_T} \quad \mathbf{0} \quad \mathbf{0}]$$

where

$\mathbf{k}_{x_T}$: displacement feedback gain vector

$\mathbf{k}_{\dot{x}_T}$: velocity feedback gain vector

$\mathbf{k}_{f_T}$: force feedback gain vector

The value of reference gain vector $\mathbf{k}_r$ depends on the controlled variable:

$\mathbf{k}_r = \mathbf{k}_{x_T}$ if the controlled variable is displacement $\mathbf{x}_T$

$\mathbf{k}_r = \mathbf{k}_{\dot{x}_T}$ if the controlled variable is velocity $\dot{\mathbf{x}}_T$

$\mathbf{k}_r = \mathbf{k}_{f_T}$ if the controlled variable is force $\mathbf{f}_T$

Computation of Mass, Damping, and Stiffness Matrices

The system mass, damping, and stiffness matrices are composed of various sub-matrices, each defined with respect to different origins. In the following sections, these sub-matrices are derived.

Table Mass Matrix $\mathbf{M}_T$

Usually mass and inertia are given at the center-of-gravity (CG) of the table, so the table mass matrix at the CG, $\mathbf{M}_T^G$, is created simply by populating the diagonal of the matrix with mass and inertia values. However, the origin of the coordinate system may not coincide with the CG, so $\mathbf{M}_T^G$ must be transformed into $\mathbf{M}_T$, the table mass matrix at the origin (here, superscript “G” denotes CG, and lack of a superscript denotes table origin):

$$\mathbf{M}_T = \mathbf{F}_G^T \mathbf{M}_T^G \mathbf{A}_T^G$$

where $\mathbf{A}_T^G$ is a matrix that transforms acceleration from the origin to the CG, and $\mathbf{F}_G^T$ is a matrix that transforms force from the CG to the origin.

The acceleration transform matrix $\mathbf{A}_T^G$ derives from the acceleration transform below, where $\mathbf{R}_T^W$ is the transpose of Euler rotation matrix $\mathbf{R}_T^W$ and $[x_{TG} \ y_{TG} \ z_{TG}]^T$ is the location of the CG relative to the origin:

$$\begin{aligned} \begin{bmatrix} \ddot{x}_G \\ \ddot{y}_G \\ \ddot{z}_G \\ \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{bmatrix} &= \begin{bmatrix} \ddot{x}_T \\ \ddot{y}_T \\ \ddot{z}_T \\ \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{bmatrix} + \begin{bmatrix} \mathbf{R}_T^W \begin{bmatrix} \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{bmatrix} \times \begin{bmatrix} x_{TG} \\ y_{TG} \\ z_{TG} \end{bmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \ddot{x}_T \\ \ddot{y}_T \\ \ddot{z}_T \end{bmatrix} + \mathbf{R}_T^W \begin{bmatrix} 0 & z_{TG} & -y_{TG} \\ -z_{TG} & 0 & x_{TG} \\ y_{TG} & -x_{TG} & 0 \end{bmatrix} \cdot \begin{bmatrix} \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \ddot{x}_G \\ \ddot{y}_G \\ \ddot{z}_G \\ \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{R}_T^W \begin{bmatrix} 0 & z_{TG} & -y_{TG} \\ -z_{TG} & 0 & x_{TG} \\ y_{TG} & -x_{TG} & 0 \end{bmatrix} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \ddot{x}_T \\ \ddot{y}_T \\ \ddot{z}_T \\ \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{bmatrix} \\ \Rightarrow \mathbf{A}_T^G &= \begin{bmatrix} \mathbf{I} & \mathbf{R}_T^W \begin{bmatrix} 0 & z_{TG} & -y_{TG} \\ -z_{TG} & 0 & x_{TG} \\ y_{TG} & -x_{TG} & 0 \end{bmatrix} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \end{aligned}$$

Note that the above transform has been simplified by omitting rotational velocity terms, which implies that the rotational velocity is assumed to be small (a reasonable assumption in most cases).

The force transform matrix $\mathbf{F}_G^T$ derives from the force transform below, where $\mathbf{R}_W^T$ is the Euler rotation matrix and $[x_{GT} \ y_{GT} \ z_{GT}]^T$ is the location of the origin relative to the CG:

$$\begin{aligned}
\begin{bmatrix} f_x \\ f_y \\ f_z \\ f_{\theta_x} \\ f_{\theta_y} \\ f_{\theta_z} \end{bmatrix} &= \begin{bmatrix} f_x \\ f_y \\ f_z \\ f_{\theta_{xG}} \\ f_{\theta_{yG}} \\ f_{\theta_{zG}} \end{bmatrix} + \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} x_{GT} \\ y_{GT} \\ z_{GT} \end{bmatrix} \right) = \begin{bmatrix} f_x \\ f_y \\ f_z \\ f_{\theta_{xG}} \\ f_{\theta_{yG}} \\ f_{\theta_{zG}} \end{bmatrix} + \left(\begin{bmatrix} 0 & z_{GT} & -y_{GT} \\ -z_{GT} & 0 & x_{GT} \\ y_{GT} & -x_{GT} & 0 \end{bmatrix} \mathbf{R}_W^T \right) \cdot \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \\
\Rightarrow \begin{bmatrix} f_x \\ f_y \\ f_z \\ f_{\theta_x} \\ f_{\theta_y} \\ f_{\theta_z} \end{bmatrix} &= \begin{bmatrix} 0 & z_{GT} & -y_{GT} \\ -z_{GT} & 0 & x_{GT} \\ y_{GT} & -x_{GT} & 0 \end{bmatrix} \mathbf{R}_W^T \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \\ f_z \\ f_{\theta_{xG}} \\ f_{\theta_{yG}} \\ f_{\theta_{zG}} \end{bmatrix} \\
\Rightarrow \mathbf{F}_G^T &= \begin{bmatrix} 0 & z_{GT} & -y_{GT} \\ -z_{GT} & 0 & x_{GT} \\ y_{GT} & -x_{GT} & 0 \end{bmatrix} \mathbf{R}_W^T \mathbf{I}
\end{aligned}$$

Table Damping Matrix $\mathbf{C}_T$

The table damping matrix $\mathbf{C}_T$ is the actuator stiffness matrix $\mathbf{C}_A$ transformed from actuator space to DOF space

$$\mathbf{C}_T = \mathbf{J}_F^{-1} (\mathbf{C}_A \otimes \mathbf{J}_V)$$

where $\mathbf{J}_F^{-1}$ is the inverse force jacobian, $\mathbf{J}_V$ is the forward velocity jacobian, and operator $\otimes$ signifies one-to-one multiplication of individual matrix elements, not matrix multiplication. Actuator damping matrix $\mathbf{C}_A$ is an $N_{ACT} \times 6$ matrix made up of six replicated actuator damping $N_{ACT} \times 1$ column vectors $\mathbf{c}_A$, i.e., $\mathbf{C}_A = [\mathbf{c}_A \ \mathbf{c}_A \ \mathbf{c}_A \ \mathbf{c}_A \ \mathbf{c}_A \ \mathbf{c}_A]$. Each element of $\mathbf{c}_A$ corresponds to an individual actuator damping c_A , given by

$$c_A = 2\zeta_A \omega_A m_A,$$

where ζ_A is actuator parasitic damping ratio (usually 6 to 8%), and ω_A is the natural frequency of the actuator pushing an effective mass of m_A through an oil spring k_A :

$$\omega_A = \sqrt{\frac{k_A}{m_A}}$$

An expression for k_A is given in the next section. The effective mass of m_A derives from the mass matrix transformed from DOF to actuator space:

$$\mathbf{M}_A = \mathbf{J}_F^{-1} \mathbf{M}_T$$

$$\mathbf{m}_A = \sum_{j=1}^3 |\mathbf{M}_A(i, j)|$$

The $N_{ACT} \times 1$ column vector of effective actuator mass $\mathbf{m}_A$ is formed by summing the absolute value of the first three columns of $\mathbf{M}_A$, i.e., only those columns corresponding to linear DOFs.

Table Stiffness Matrix $\mathbf{K}_T$

The table stiffness matrix $\mathbf{K}_T$ is the actuator stiffness matrix $\mathbf{K}_A$ transformed from actuator space to DOF space

$$\mathbf{K}_T = \mathbf{J}_F^{-1} (\mathbf{K}_A \otimes \mathbf{J}_D)$$

where $\mathbf{J}_F^{-1}$ is the inverse force jacobian, $\mathbf{J}_D$ is the forward displacement jacobian, and operator $\otimes$ signifies one-to-one multiplication of individual matrix elements, not matrix multiplication. Actuator stiffness matrix $\mathbf{K}_A$ is an $N_{ACT} \times 6$ matrix made up of six replicated actuator stiffness $N_{ACT} \times 1$ column vectors $\mathbf{k}_A$, i.e., $\mathbf{K}_A = [\mathbf{k}_A \ \mathbf{k}_A \ \mathbf{k}_A \ \mathbf{k}_A \ \mathbf{k}_A \ \mathbf{k}_A]$. Each element of $\mathbf{k}_A$ corresponds to an individual actuator stiffness k_A , given by

$$k_A = \beta \left(\frac{A_1}{L_1} + \frac{A_2}{L_2} \right),$$

where β is the bulk modulus of oil, A_1 and A_2 are piston areas, and L_1 and L_2 are oil column lengths.

Spring/Damper Specimen Matrices $\mathbf{C}_R$ and $\mathbf{K}_R$

Usually spring specimen properties are given at the point $[x_{TR} \ y_{TR} \ z_{TR}]^T$ that it attaches to the table, so the spring/damper specimen matrices $\mathbf{C}_R^R$, and $\mathbf{K}_R^R$ created simply by populating the diagonal of the matrices with damping and stiffness values. These matrices must be transformed to the table origin as follows (here, superscript “R” denotes spring/damper specimen attachment point, and lack of a superscript denotes table origin):

$$\mathbf{C}_R = \mathbf{F}_R^T \mathbf{C}_R^R \mathbf{V}_T^R$$

$$\mathbf{K}_R = \mathbf{F}_R^T \mathbf{K}_R^R \mathbf{D}_T^R$$

where $\mathbf{D}_T^R$ and $\mathbf{V}_T^R$ are matrices that transforms displacement and velocity from the origin to the specimen attachment point, and $\mathbf{F}_R^T$ is a matrix that transforms force from the attachment point to the origin.

The force transform matrix $\mathbf{F}_R^T$ is similar to that derived in a previous section:

$$\mathbf{F}_R^T = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \begin{bmatrix} 0 & z_{RT} & -y_{RT} \\ -z_{RT} & 0 & x_{RT} \\ y_{RT} & -x_{RT} & 0 \end{bmatrix} & \mathbf{R}_W^T \mathbf{I} \end{bmatrix}$$

The velocity transform matrix $\mathbf{V}_T^R$ derives from the velocity transform below, where $\mathbf{R}_T^W$ is the transpose of Euler rotation matrix $\mathbf{R}_W^T$ and $[x_{TR} \ y_{TR} \ z_{TR}]^T$ is the location of the attachment point relative to the origin:

$$\begin{aligned} \begin{bmatrix} \dot{x}_T \\ \dot{y}_T \\ \dot{z}_T \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} &= \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{z}_R \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} + \begin{bmatrix} \mathbf{R}_T^W \left(\begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \times \begin{bmatrix} x_{TR} \\ y_{TR} \\ z_{TR} \end{bmatrix} \right) \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{z}_R \end{bmatrix} + \mathbf{R}_T^W \begin{bmatrix} 0 & z_{TR} & -y_{TR} \\ -z_{TR} & 0 & x_{TR} \\ y_{TR} & -x_{TR} & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \dot{x}_T \\ \dot{y}_T \\ \dot{z}_T \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{R}_T^W \begin{bmatrix} 0 & z_{TR} & -y_{TR} \\ -z_{TR} & 0 & x_{TR} \\ y_{TR} & -x_{TR} & 0 \end{bmatrix} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{z}_R \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \\ \Rightarrow \mathbf{V}_T^R &= \begin{bmatrix} \mathbf{I} & \mathbf{R}_T^W \begin{bmatrix} 0 & z_{TR} & -y_{TR} \\ -z_{TR} & 0 & x_{TR} \\ y_{TR} & -x_{TR} & 0 \end{bmatrix} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \end{aligned}$$

The displacement transform matrix $\mathbf{D}_T^R$ is simply the transpose of the force transform matrix $\mathbf{F}_T^R$:

$$\mathbf{D}_T^R = \mathbf{F}_R^T'$$

The force transform is used instead of the true displacement transform because the latter is an identity matrix plus an offset vector rather than a pure matrix. Because of the direct relationship between displacement and force jacobians (they are transposes of each other), it is reasonable to use the force transform as a substitute.

Modal Specimen Matrices $\mathbf{M}_S^T$, $\mathbf{C}_S^T$, and $\mathbf{K}_S^T$

Usually modal specimen properties are given at the CG of the modal specimen, so the modal specimen matrices $\mathbf{M}_S$, $\mathbf{C}_S$, and $\mathbf{K}_S$ created simply by populating the diagonal of the matrices with mass, damping, and stiffness values. These matrices must be transformed to the table origin origin. The derivation of the transforms is similar to that previous sections, so only the results are given below (here, superscript “T” denotes table origin, and lack of a superscript denotes modal specimen CG):

$$\mathbf{M}_S^T = \mathbf{F}_S^T \mathbf{M}_S \mathbf{A}_T^S$$

$$\mathbf{C}_S^T = \mathbf{F}_S^T \mathbf{C}_S^S \mathbf{V}_T^S$$

$$\mathbf{K}_S^T = \mathbf{F}_S^T \mathbf{K}_S^G \mathbf{D}_T^S$$

where

$$\mathbf{F}_S^T = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \begin{bmatrix} 0 & z_{ST} & -y_{ST} \\ -z_{ST} & 0 & x_{ST} \\ y_{ST} & -x_{ST} & 0 \end{bmatrix} & \mathbf{R}_W^T \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{A}_T^S = \begin{bmatrix} \mathbf{I} & \mathbf{R}_T^W \\ \mathbf{0} & \begin{bmatrix} 0 & z_{TS} & -y_{TS} \\ -z_{TS} & 0 & x_{TS} \\ y_{TS} & -x_{TS} & 0 \end{bmatrix} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{V}_T^S = \begin{bmatrix} \mathbf{I} & \mathbf{R}_T^W \\ \mathbf{0} & \begin{bmatrix} 0 & z_{TS} & -y_{TS} \\ -z_{TS} & 0 & x_{TS} \\ y_{TS} & -x_{TS} & 0 \end{bmatrix} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{D}_T^S = \mathbf{F}_S^{T'}$$

Computation of Frequency Response Functions (FRFs)

FRFs for every input DOF to every output DOF can be computed in two steps using MATLAB functions:

- 1) Convert the state-space equations into a matrix of Laplace transfer function polynomials using *ss2tf*.
- 2) Evaluate the Laplace transfer function polynomials on a grid of frequencies using *freqs*.